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THE ELEMENTS
OF PLANE AND SOLID
MENSURATION

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MENSURATION

WITH COPIOUS EXAMPLES

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PREFACE

THE present work is an attempt to provide an adequate text-book for a branch of Mathematics which, though recognized in several examinations, has hardly received the attention it deserves. The late Professor Todhunter's small work on Mensuration is inadequate to present needs, chiefly because it treats the subject solely as a practical application of Arithmetic, whereas it is far better to regard it as mainly, indeed, dependent on Arithmetic, but largely also on Geometry and Trigonometry.

I have kept steadily in view the requirements of present examinations, especially the Sandhurst Competitive and the Oxford Local. I have laid great stress on the formulæ, and made their application clear by illustrative examples worked right through. While I trust the work will be found useful for those preparing for examinations, I hope that it may also claim to be a full and worthy exposition of the subject. The requirements of space have compelled me to pass lightly over a few points, such as the prismoid, wedge, and solid ring, also the proof of some of the formulæ used in Book II. With these exceptions I believe the subject is complete as far as it goes. I have spent much labour in endeavouring to ensure correctness in the answers. However, I cannot expect to have entirely avoided mistakes, which I shall be glad to have pointed out to me.

F. G. B.

11, Museum Villas, Oxford,
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INTRODUCTORY

1. **MENSURATION** is the art of measuring. It is one of the practical applications of Mathematics, being the carrying out in detail of principles established by Arithmetic, Geometry, and Trigonometry. These principles are applied to the solution of three classes of problems; namely, the determination from certain *data* of:

(1) The Lengths of lines.

(2) The Areas of surfaces, plane or otherwise.

(3) The Volumes of solids.

2. The *data* are (1) straight lines, (2) straight lines and angles. The student must be prepared for problems of the reverse order, in which the straight lines and angles themselves have to be determined. The introduction of angles, and the results about them established by Trigonometry, adds greatly to the scope of Mensuration in determining lengths and areas.

3. Mensuration has two branches, plane and solid. Plane Mensuration determines the lengths of lines, and the areas of plane figures; Solid Mensuration determines the areas of surfaces, and the volumes of solid figures. The plane figures treated of in the first part are (1) rectilinear figures, which are called *triangles*, *quadrilaterals*, or *polygons* (regular or irregular), according as they have three, four, or more

sides; (2) the circle. Every chapter is divided into two sections, the first treating of lengths, the second of areas. The solid figures treated of in the second part are only the simpler forms; i.e. the *parallelepiped*, the *prism*, the *pyramid*, the *cylinder*, the *cone*, and the *sphere*. Each chapter is divided into two sections, the first treating of surfaces, the second of volumes.

4. The student should carefully bear in mind that Mensuration is a practical application of rules. These rules are conveniently stated as *Formulæ*, and are placed prominently at the head of every section. Many of these formulæ should be learnt by heart, and their ready application is the great object to be aimed at.

Besides a knowledge of Arithmetic, Euclid, and Trigonometry, some knowledge of Algebra is required to solve certain examples; while several more assume a knowledge of logarithms.

5. This is the best place to say a word about the *Tables* used. The ordinary Arithmetical tables are appended.

(1) LENGTH (LINEAR MEASURE).

12	inches	make	foot.
3	feet	„	yard.
5½	yards	„	rod, pole, or perch.
40	poles	„	furlong.
8	furlongs	„	mile.

(2) SURFACE (SQUARE MEASURE).

144	sq. inches	make	1 sq. foot.
9	sq. feet	„	1 sq. yard.
30¼	sq. yards	„	1 sq. pole or perch.
40	poles	„	1 rood.
4	roods	„	1 acre.
640	acres	„	1 sq. mile.

(3) SOLIDITY (CUBIC MEASURE).

1728 cubic inches make 1 cubic foot.
27 cubic feet „ 1 cubic yard.

(1) is used in measuring lines, the unit being the inch.

(2) is used in measuring areas of surfaces, whether plane or otherwise. The unit is the square inch; i.e. a plane figure which is an inch long and an inch broad.

(3) is used in measuring solids. The unit is the cubic inch; i.e. a solid figure which is an inch either way, in breadth, length, and thickness.

The area of a field is usually given in acres, roods, and perches (a. r. and p.). If an area is found in square yards, it may be reduced to acres, roods, and perches, or, since there are 4840 square yards in an acre, it may be expressed in acres and yards. In practice the *linear* measurements of fields are taken by a *chain* 22 yards in length, and divided into 100 links. The square measurements are taken by the *square chain*, 484 square yards in area, and divided into 10,000 square links. The student will observe that 10 square chains make an acre. Thus we may add to the linear measurements:

100 links make 1 chain.
1 chain = 22 yards.

And to the square measurements:

10,000 sq. links make 1 sq. chain.
1 sq. chain = 484 sq. yards.
10 sq. chains make 1 acre.

6. The practical value of using the chain is obvious, since it substitutes for the cumbrous linear and square measures a new measure based on the number 10, thus lending itself to decimal notation. This principle has been

carried out to its fullest extent in the *Metric system* established in France. The tables of linear, square, and solid measure employed are as follows:

(1) Linear Measure.

10 millimetres = 1 centimetre.
10 centimetres = 1 decimetre.
10 decimetres = 1 metre.
10 metres = 1 dekametre.
10 dekametres = 1 hectometre.
10 hectometres = 1 kilometre.

The linear unit is the *metre*, which = 39.37 inches.

(2) Square Measure.

100 centiares = 1 are.
100 ares = 1 hectare.

The square unit is the *are*, which = 119.6 square yards. The are is a square each side of which is 10 metres.

(3) Solid Measure.

10 decisteres = 1 stere.
10 steres = 1 dekastere.

The solid unit is the stere, which = 61,027 cubic inches. The stere is a cube every dimension of which is 1 metre.

It will be observed that the Latin prefixes *deci-*, *milli-*, are always used for the divisions of the unit, and the Greek prefixes *deka-*, *hecto-*, *kilo-*, for the multiples of the unit.

Book I.

CHAPTER I.—TRIANGLES

Sec. I.—Solution of Triangles

[FORMULÆ:

A, B, C are the angles; a, b, c the sides opposite to them; s the semi-perimeter.

(1) $A + B + C = 180°$.

(2) $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

(3) $a^2 = b^2 + c^2 - 2bc \cos A$.

(4) $\tan \dfrac{B - C}{2} = \dfrac{b - c}{b + c} \cot \dfrac{A}{2}$.

(5) $\sin \dfrac{A}{2} = \sqrt{\dfrac{(s - b)(s - c)}{bc}}$; $\cos \dfrac{A}{2} = \sqrt{\dfrac{s(s - a)}{bc}}$;

$$\tan \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}.$$

(6) If C be a right angle, $c^2 = a^2 + b^2$.]

7. The first problem in Mensuration is the determination of the lengths of the sides of triangles from given data. This is so fully treated of in Trigonometry, under the head "Solution of Triangles," that it is sufficient to recapitulate the main results. It is shown that if three of the six elements of a triangle are given, the remaining three can be determined, except in the case where the three angles are given, when only the ratios of the sides can be determined. Setting this aside, there are four cases to consider.

CASE I.—*When two angles and a side are given* (e.g. A, B, and a).

$$C = 180^\circ - A - B. \quad \text{(Formula 1.)}$$

$$b = \frac{a \sin B}{\sin A}, \text{ and } c = \frac{a \sin C}{\sin A}. \quad \text{(Formula 2.)}$$

Hence b, c, C are determined.

CASE II.—*When two sides and the included angle are given* (e.g. b, c, and A).

If the third side *only* is to be determined, we can use the formula (3): $a^2 = b^2 + c^2 - 2bc \cos A$.

N.B.—This formula is not adapted to logarithmic computation.

If the other angles are to be found, we have:

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} \quad . \quad . \quad \text{(Formula 4.)}$$

$$\text{and } \frac{B+C}{2} = 90^\circ - \frac{A}{2} \quad . \quad . \quad . \quad . \quad . \quad \text{(Formula 1.)}$$

When B and C are known, a can be determined by Formula 2.

CASE III.—*When two sides and the opposite angle are given* (e.g. a, b, and A. Ambiguous case).

If the third side only is to be determined,

$$a^2 = b^2 + c^2 - 2bc \cos A. \quad \text{(Formula 3.)}$$

This gives a quadratic for determining c. If both the roots are real and positive, two triangles can be formed from the given parts. If one is positive and the other negative, there is only one triangle.

If the other angles are to be determined, we have:

Corresponding to one value of sin B, there are two values of B, one acute and the other obtuse. The latter will only be applicable if $b > a$, for otherwise B will not be the greatest angle in the triangle.

CASE IV.—*When the three sides are given* (i.e. a, b, c).

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}};\quad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}};$$

$$\text{[illegible]}\frac{\text{[illegible]}}{2}\text{[illegible]}\sqrt{\frac{\text{[illegible]}-b)(s-c)}{s(s-a)}}.\quad \text{(Formula 5.)}$$

And similar formulæ hold for the other angles.

8. Three particular cases should be noticed.

(a) The right-angled triangle (C the right angle).

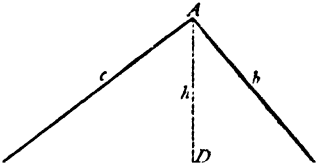
Here $a^2 + b^2 = c^2$. (Euclid i. 47.)

Also the ratio of any one side to any other side is some trigonometrical ratio of both A and B. Thus, if a and b are given, we have:

$$\tan A = \cot B = \frac{a}{b},$$

so that A and B are at once determined.

By the help of this property, we may determine the *altitude* of any triangle; i.e. the perpendicular AD from the vertex on the base. In the right-angled triangle ABD we have $AD = c \sin B$, and in the right-angled triangle ACD we have [illegible]



The altitude is usually denoted by the letter h.

(*b*) *The isosceles triangle* ($b = c$).

If the two equal sides (b, c) and the included angle A are given, then

$$B = C, \text{ and } B + C = 180° - A.$$

$$\therefore \ B = C = 90° - \frac{A}{2}.$$

$$a = \frac{b \sin A}{\sin B}.$$

Hence B, C, a, are determined.

Notice that in an isosceles triangle the perpendicular from the vertex bisects the base.

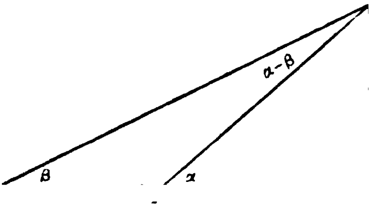
(*c*) *The equilateral triangle* ($a = b = c$).

Since $A = B = C$, each angle $= 60°$.

Hence the triangle is completely determined by the length of the side.

Examples.—The principal application of the above formulæ is to the measurement of heights and distances. Heights are usually measured in one of the two ways indicated in the first two examples.

(1) A lighthouse appears to a man in a boat to subtend an angle β. After rowing a yards directly towards the lighthouse, he finds it subtends an angle α. Find the height of the lighthouse.



Let AB be the lighthouse, C and D the two points at which the angles β and α were observed.

The angle CAD obviously $= \alpha - \beta$.

Now $AB = AD \sin \alpha$,

$$\text{and } \frac{AD}{\sin \beta} = \frac{CD}{\sin (\alpha - \beta)}. \quad \therefore AD = \frac{a \sin \beta}{\sin (\alpha - \beta)}.$$

$$\therefore AB = \frac{a \sin \alpha \sin \beta}{\sin (\alpha - \beta)}. \quad \text{Answer.}$$

Let $\alpha = 75^\circ$, $\beta = 60^\circ$, $a = 30$ ft.

$$\therefore \text{ height of lighthouse} = \frac{30 \sin 60^\circ \sin 75^\circ}{\sin 15^\circ} \text{ feet.}$$

$$= \frac{30 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{15\sqrt{3}(\sqrt{3}+1)}{\sqrt{3}-1} = \frac{15\sqrt{3}(\sqrt{3}+1)^2}{3-1}.$$

$$= \frac{15\sqrt{3}(4+2\sqrt{3})}{2} = 15(2\sqrt{3}+3) \text{ feet.}$$

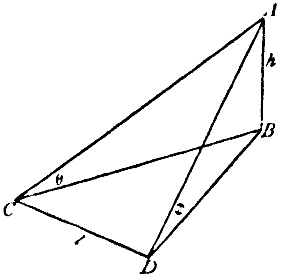
$$\sqrt{3} = 1{\cdot}732. \qquad \therefore 2\sqrt{3}+3 = 6{\cdot}464$$

$$\begin{array}{r} 6{\cdot}464 \\ 15 \\ \hline 32320 \\ 6464 \\ \hline 96\ 960 \end{array}$$

∴ height of lighthouse = 97 feet (nearly). Answer.

(In this example Case I. in the solution of triangles is used. The angles at *C* and *D* are called the *angles of elevation* of the point *A*. If they were observed from the top of the tower *A*, the same angles would be called *angles of depression* of *C* and *D*.)

(2) A measured line is drawn from a point on a horizontal plane in a direction at right angles to the line joining that point to the base of a tower standing on the plane. The length of the line is l, and the angles of elevation of the tower from the two ends of it are θ and ϕ. Find the height of the tower.



Let *AB* be the tower, and *CD* the measured line; then *BCD* is a right angle. Let h stand for *AB*.

From right-angled triangle ACB,

$$CB = h \cot \theta,$$

and from right-angled triangle ADB,

$$DB = h \cot \phi.$$

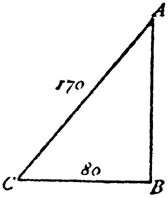
But $$DB^2 - CB^2 = CD^2 = l^2.$$

$$\therefore\ h^2 (\cot^2 \phi - \cot^2 \theta) = l^2.$$

$$h = \frac{l}{\sqrt{\cot^2 \phi - \cot^2 \theta}}.\quad \text{Answer.}$$

(This example well illustrates the use of right-angled triangles. The student will notice that all the points are not in one plane; B, C, D being in the horizontal plane, and A above it.)

(3) A string 170 feet is stretched from the top of a tower, and reaches the ground 80 ft. from its base. Find the height of the tower.



In the right-angled triangle ABC,

$$AB^2 = AC^2 - CB^2.$$

$$\therefore\ AB = \sqrt{170^2 - 80^2}\text{ feet.}$$

$$= \sqrt{250 \cdot 90} = 10 \times 5 \times 3.$$

$$= 150\text{ feet.}\quad \text{Answer.}$$

EXAMPLES ON THE SOLUTION OF TRIANGLES

(a) Right-angled Triangle

1. A ladder 8 ft. 4 in. long is set up against a wall, 2 ft. 4 in. from it, and is then observed to reach to within 2 ft. of the top. Required the height of the wall.

2. A line stretched from the top of a tower to a station 28 ft. from its base measures 100 ft. Find the height of the tower.

3. A ladder 65 ft. long has its base planted in a street, and reaches on one side to a height of 63 ft., and on the other to a height of 52 ft. Required the breadth of the street.

4. A footpath runs from A to B round the right angle ACB. At what distance from C in AC must a point D be taken, so that a short cut from D to B may save 72 yds.? Given $BC = 126$ yds.

5. A place A is 42 miles east of another place B, and 40 miles north of a third place C. Find the distance between B and C.

6. A person observes a tower to the north which he knows to be a feet high, at an elevation θ, and another to the east, b feet high, at an elevation ϕ. Find the distance between the two towers. Example: $\theta = 30^\circ$, $\phi = 45^\circ$, $a = 160$ ft., $b = 40$ ft.

7. A measured line is drawn from a point in a horizontal plane in a direction at right angles to the line joining that point to the base of a tower standing on the plane. The angles of elevation of the tower from the two ends of the measured line are 30° and 18°. Find the height of the tower in terms of l, the length of the measured line. (*Sandhurst.*)

8. A lighthouse appears to a man in a boat 300 yds. from its base to subtend an angle of 6° 20′ 24.7″. Find in feet the height of the lighthouse, having given *L* tan 6° 20′ = 9.0452836. Diff. for 1′ = .0011507, log 3 = .4771213. (*Sandhurst.*)

9. Find the height of a tower which subtends an angle of 56° 18′ at the distance of 100 ft. (Answer correct to four decimal places of a foot.) Given *L* tan 56° 18′ = 10.1759281, log 14994 = 4.1759175, *D* = 289.

10. Find the breadth of a river from the further bank of which a tower 140 ft. high subtends an angle of 71° 34′. Given log 2 = .30103, log 7 = .84510, *L* cot 71° 34′ = 10.47716, log 42004 = 4.62329.

11. Find to five decimal places of a foot the length of a wire stretched from the top of a flagstaff 75 ft. high, and making an angle 77° 43′ with the ground. Given log 2 = .3010300, log 3 = .4771213, *L* cosec 77° 43′ = 10.0100577, log 76.757 = 1.8851180, *D* = 57.

(b) Any Triangle

12. Find to three places of decimals of a foot the altitude of an isosceles triangle whose sides are each 17 feet, and whose base is 9 ft. 6 in.

13. Find the altitude of a triangle whose sides are 6 ft. 10 in. and 9 ft. 8 in., and whose base is 8 ft. 6 in.

14. A tower was observed to subtend angles of 45° and 30° at two points in the same horizontal straight line drawn through the base of the tower. If the distance between these two points is 183 ft., find the height of the tower to the nearest foot.

15. *A*, *B*, *C* are three points in a straight line on a level piece of ground. A vertical pole is erected at *C*; the angle of elevation of its top, as observed from *A*, is 5° 30′, and, as observed from *B*, is 10° 45′, the distance from *A* to *B* being 100 yards. Find the distance *BC*, and the height of the pole. (*Sandhurst.*)

16. From the top of a tower, whose height is 100 ft., the angles of depression of two small objects on the plain below, which are in the same vertical plane as the tower, are observed, and found to be 45° and 30°. Find the distance between them to two places of decimals. (*Sandhurst.*)

17. The front of a house subtends an angle 45° from an opposite window. Given the height of the window from the ground (= 10 ft.), and the breadth of the street (= 30 ft.), show how to find the height of the house.

18. An object is observed from two points A and B, both in the horizontal straight line passing directly beneath it at C. The angle of elevation at B is double that at A. Given $AB = 50$ ft., and $BC = 14$ ft., find the height of object.

19. If a tower stands at the foot of a hill whose inclination to the horizon is 9°, and if from a point 100 ft. up the hill the tower subtends an angle of 54°, find its height. (*Sandhurst.*)

20. Three points, A, B, C, are known by an observer to be in a straight line. His distance from A is 100 ft., and from B 50 ft. If AB subtends at his eye an angle of 60°, and BC an angle of 15°, show that his distance from C is $\frac{100\sqrt{2}}{\sqrt{3}+1}$ ft.

21. Find the distance between two buoys, which are $\frac{1}{4}$ mile and $\frac{5}{8}$ mile distant respectively from a point on the shore, at which point they subtend an angle 49° 27′ 30″, whose cosine is given as .65.

22. A flagstaff 30 ft. long leans at an angle of 60°; a cord 27 ft. long is stretched from the top so as to exactly reach the ground. At what distance from the flagstaff can it do this? Show that there are two solutions.

23. Two straight railroads are inclined at an angle of 20° 16′. At the same instant, from their point of intersection, an engine starts along each line. One travels at the rate of 20 miles an hour. At what rate must the other

travel so that after 3 hours the engines shall be at a distance from each other of 30 miles? Show that the question admits of two solutions. (*Sandhurst.*)

24. A straight pine-tree leans due N.E. At a yds. distance due N.E. it subtends the angle α; at a yds. distance due S.W. the angle β. Show that the height of the pine-tree is:

$$\sin(\alpha+\beta)\ \sqrt{\sin(\alpha-\beta)+4\sin\alpha\sin\beta}.$$

What does this value become if α and β are complementary?

25. An observer is situated in a boat vertically beneath the centre of the roadway of a suspension bridge. He finds that its length subtends at his eye an angle α. At a measured distance y down stream, at a point immediately opposite the centre of the roadway, he finds it subtends an angle β. Supposing the surface of the river horizontal, find an expression for the length of the roadway, and its altitude above the river. (*Sandhurst.*)

26. Measured from the ground, the elevation of a steeple is α; at a distance a vertically above the first place of observation the angular elevation is β. Find the height of the steeple, and its distance from the place of observation. Example: Let $\alpha = 45°$, $\beta = 30°$, then the height will be $\frac{a}{2}(3+\sqrt{3})$, and the distance $\frac{a}{2}(3+\sqrt{3})$.

27. A mountain at 400 yds. distance from its base was seen at an elevation of 44°. On advancing to the base along level ground, its elevation was 57°. Find the height of the mountain in feet. Given $\log 12 = 1.0791812$, $L \sin 13° = 9.3520880$, $L \sin 44° = 9.8417713$, $L \sin 57° = 9.9235914$, $\log 31078 = 4.4924531$, $D = 140$.

28. A person wanting to calculate the perpendicular height of a cliff, takes its angular altitude, 12° 30′, and then measures 950 yds. in a direct line towards the base, when he is stopped by a river; he then takes a second altitude, and finds it 69° 30′. Find the height of the cliff. Given

$\log 5 = .6989700$, $L \sin 12^\circ 30' = 9.3353368$, $\log 19 = 1.2787536$, $L \cos 33^\circ = 9.9235914$, $\log 2296 = 3.3610566$, $L \cos 20^\circ 30' = 9.9715876$. (*Sandhurst.*)

29. Derwentwater Lake is 238 ft. above the sea-level. From a boat on the lake the elevation of Skiddaw is observed to be 7°. After rowing a mile straight towards the mountain, the elevation is observed to be $9^\circ 4' 30''$. Find the height in feet of the highest point of Skiddaw visible from the lake. Given $\log 528 = 2.72263$, $L \sin 7^\circ = 9.08589$, $L \sin 9^\circ 4' 30'' = 9.19791$, $L \sin 2^\circ 4' 30'' = 8.55880$, $\log 28031 = 4.44763$.

30. The elevation of an object, when viewed from a point A to the south of it, is θ; when viewed from a point B to the north of it, it is ϕ. Given $AB = a$, find the height of the object. Example: Let $\theta = 52^\circ$, $\phi = 34^\circ$, $a = 100$ ft. Given $L \sin 52^\circ = 9.8965321$, $L \sin 34^\circ = 9.7475617$, $L \sin 86^\circ = 9.9989408$, $\log 44.172 = 1.6451471$, $D = 98$.

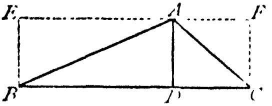
Sec. II.—Area of Triangles

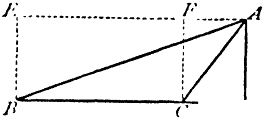
[FORMULÆ for area of triangle (S):

(1) $S = \frac{1}{2}$ base × altitude.

(2) $S = \dfrac{bc \sin A}{2}$

(3) $S \quad \sqrt{s(s-a)(s-b)(s-c)}$.]





9. To prove the formulæ for the area of a triangle.

(1) *Area of triangle* = $\frac{1}{2}$ *base* × *altitude.*

To determine the area of any triangle ABC, through A draw EAF (or EFA) parallel to the base, and draw CF, BE, parallel to AD, the altitude.

Then $EBCF$ is a parallelogram on the same base as ABC, and between the same parallels EF, BC.

∴ area of $ABC = \frac{1}{2}\, EBCF$. (Euclid, i. 41.)

$= \frac{1}{2}\, EB \,.\, BC$. (See chap. ii.)

$= \frac{1}{2}\, BC \,.\, AD$.

$= \frac{1}{2}$ base × altitude. Q.E.D.

Or the result may be obtained thus:

$ABD = \frac{1}{2}\, ED$, and $ACD = \frac{1}{2}\, AC$.

∴ whole or difference $ABC = \frac{1}{2}$ whole or difference $EBCF$.

N.B.—In a triangle any side may be taken for base, and the perpendicular let fall from the opposite angle is the altitude.

It is to be observed that we can only prove the expression for the area of a triangle by assuming that for the area of a rectangle. This will be fully elucidated in the next chapter.

(2) Since area of triangle $= \frac{1}{2} a \times$ altitude, and altitude $= c \sin B$,

$$\therefore \text{ area} = \frac{ac \sin B}{2}. \quad \text{Q.E.D.}$$

This expression will be found equivalent to

$$\frac{ab \sin C}{2}, \text{ and } \frac{bc \sin A}{2}.$$

(3) Again, area $= \dfrac{bc \sin A}{2}$

$$= bc \,.\, \sin \frac{A}{2} \,.\, \cos \frac{A}{2}.$$

Substituting the values given in sec. i., we have :

$$\text{Area} = bc \,.\, \sqrt{\frac{(s-b)(s-c)}{bc}} \,.\, \sqrt{\frac{s(s-a)}{bc}}.$$

$$= \sqrt{s(s-a)(s-b)(s-c)}. \quad \text{Q.E.D.}$$

10. *Particular cases.*

(1) In a *right-angled triangle* ($C =$ right angle),

$$\text{Area} = \frac{ab}{2}.$$

(2) In an *equilateral* triangle (side $= a$),

$$\text{Area} = \frac{a^2}{2} \sin 60° = \frac{\sqrt{3}a^2}{4}.$$

Examples.—(1) Find the area of a triangular field whose sides are 124 yds. 2 ft., 238 yds. 1 ft., and 341 yds. respectively; and its rent at £1 10*s.* per acre.

Here we use formula (3).

$$a = 374 \text{ ft.}, \; b = 715 \text{ ft.}, \text{ and } c = 1023 \text{ ft.}$$

$$\therefore s = \tfrac{1}{2} \times 2112 = 1056 \text{ ft.}$$

$$\text{Area} = \sqrt{1056\,(1056-374)\,(1056-715)\,(1056-1023)}.$$

$$= \sqrt{\underset{\underset{\cancel{16}}{\cancel{352}}}{\cancel{1056}} \times \underset{\cancel{22}}{\cancel{682}} \times \underset{\cancel{31}}{\cancel{341}} \times \underset{3}{\cancel{33}}}$$

$$= 11 \times 3 \times 31 \times 22 \times 4 \text{ sq. ft.}$$

$$= \frac{\cancel{11} \times \cancel{3} \times 31 \times \overset{2}{\cancel{22}} \times 4 \times 4}{\underset{3}{\cancel{9}} \times \cancel{121}} \text{ sq. poles.}$$

$$= \frac{62 \times 16}{3} = 330\tfrac{2}{3} \text{ sq. poles.}$$

$$= 2 \text{ a. } 0 \text{ r. } 10\tfrac{2}{3} \text{ p. Answer.}$$

Rent for 2 a. $= £3$.

$$\text{Rent for } 10\tfrac{2}{3} \text{ p.} = 10\tfrac{2}{3} \times \frac{30}{4 \times 40} \, s. = \frac{\overset{2}{\cancel{32}} \times \overset{3}{\cancel{30}}}{\cancel{3} \times \cancel{4} \times \underset{\cancel{4}}{\cancel{40}}}.$$

$$= 2s.$$

Rent $= £3\ 2s.$ Answer.

(Notice here how the labour of multiplication and extracting the square root is avoided by considering each factor of the square root separately.)

(2) The sides of a triangular field are 20 and 30 chains, and included angle is $72^\circ 15'$. Find the area.

Here we use formula (2). Since $\sin 72^\circ 15'$ must be found from the tables, we shall work most conveniently by logarithms. The logarithms we require are $\log 3 = \cdot 4771213$, $L \sin 72^\circ 15' = 9\cdot 9788175$.

$$S = \frac{bc \sin A}{2}.$$

$$= \frac{20 \cdot 30}{2} \sin 72^\circ 15' = \overset{\text{sq. chains.}}{300} \sin 72^\circ 15'.$$

$$\therefore \log S = \log 3 + 2 + L \sin 72^\circ 15' - 10.$$

$$\begin{array}{r} 2\cdot4771213 \\ 9\cdot9788175 \\ \hline 12\cdot4559388 \\ 10 \\ \hline 2\cdot4559388 \\ \hline \end{array}$$

From the tables we find:

$$\log 285{\cdot}71 = 2{\cdot}4559254,\ \text{diff.} = 152.$$
$$\therefore\ 152 : 134 :: 1 : {\cdot}88.$$
$$S = 285{\cdot}7188 \text{ sq. chains.}$$
$$= 28{\cdot}57188 \text{ acres.}$$
$$= 28 \text{ a. } 2 \text{ r. } 11\tfrac{1}{2} \text{ p.} \quad \text{Answer.}$$

(3) The area of an equilateral triangle is 25 sq. in. Find its perimeter.

Let a be the side of the triangle; then we have to find the value of $3a$, given that $\frac{\sqrt{3}a^2}{4} = 25$.

$$a^2 = \frac{100}{\sqrt{3}} \text{ sq. in.}$$

$$9a^2 = \frac{900}{\sqrt{3}} = 300\sqrt{3} \text{ sq. in.}$$

Substituting for $\sqrt{3}$, and extracting the square root, we shall find:

$$3a = 22{\cdot}8 \text{ in. very nearly.}$$

(N.B.—$\sqrt{3}$ can usually be taken as $1{\cdot}732$.)

EXAMPLES ON THE AREA OF TRIANGLES

1. The base of a triangle is 32 yds. 1 ft. 5 in., and the altitude is 19 yds. 2 ft. 10 in. Find the area.

2. The area of a triangle is 1 a. 2 r., and the altitude is 3 chains. Find the base.

3. Two sides of a triangle are 4 ft. and 6 ft., and the included angle is 60°. Find the area of the triangle. (*Sandhurst.*)

4. Two sides of a triangle which contains 1008 sq. ft. are to one another as 4 to 7, and the included angle is 30°. Find the two sides.

5. The sides of a triangle are 10 in., 17 in., and 21 in. Find the area.

6. Find the area in acres of a triangle whose sides are 25, 20, 15 chains. (*Sandhurst.*)

7. Find the area of the triangle whose sides are 13, 14, 15. (*Sandhurst.*)

8. Find the area of the triangle whose sides are 21, 20, 29. (*Sandhurst.*)

9. The sides of a triangle are 6 in., 7 in., and 9 in. Find the area correct to three decimal places of a square inch.

10. The two angles of a triangle are 45° and 75°, and the side opposite the third angle is 1 ft. Find the area to two decimal places of a square inch.

11. The two sides of a right-angled triangle are to each other as 5 to 12, and the area is 10 sq. ft. 30 in. Find the hypotenuse.

12. The area of a right-angled triangle is 1 sq. ft. 66 in., and the hypotenuse is 2 ft. 5 in. Find the sides.

13. Find to three places of decimals the hypotenuse of a right-angled isosceles triangle equal in area to the equilateral triangle whose side is 4 in.

14. In an isosceles triangle one of the equal sides is to the third side as 5 to 6. The perimeter being 80 ft., find the area.

15. Find the area of an equilateral triangle whose side is 10 in.; and if the area be 75 sq. in., find its side to two decimal places.

16. The sides of a triangle are 34 ft., 65 ft., and 93 ft. Deduce from the area the value of the perpendicular dropped on to the longest side from the opposite angle.

17. Find the area of a triangular field, one side of which is 1 fur. 20 po., if the perpendicular dropped on it from the opposite angle is 148 yds. 2 ft.

18. The sides of a triangular field are respectively 10 chains, 8 chains, and 12 chains. The chain being 22 yds., find the acreage of the field, and the perpendicular distance of the longest side from the opposite corner. (*Sandhurst.*)

19. The area of a triangular field is 6 a. 2 r. 8 p., and the perpendicular from one angle on the base is 524 links. Find the length of the base in chains. (*Sandhurst.*)

20. In a triangular field straight lines are drawn from the angular points to a point in the field, dividing it into three parts. Given that the sides of the field are 39, 42, and 45 feet respectively, and that the parts into which it is divided are in the proportion 39 : 20 : 113 respectively, find the sum of the perpendiculars from that point.

21. Two sides of a triangular field containing an obtuse angle are 110 and 220 yds. Find the length of the third side, that the field may contain exactly an acre. (*Sandhurst.*)

22. The sides of a triangular field are 350, 440, and 750 yds. The field is let for £31 10*s.* per year; what is the price per acre? (*Sandhurst.*)

23. Which pays the most rent of two adjacent triangular fields, which have one side in common (length = 200 yds.), if the angles adjacent to the common side are in one field 30° and 45°, and in the other 15° and 45°? The first field is let for £1 10*s.* 3*d.* per acre, and the second for £2 12*s.* 11¼*d.* per acre.

24. Cost of running a fence round a right-angled triangular enclosure whose area is 33 a. 150 sq. yds., and whose sides are as 5 : 12, at 10*d.* per yd.

25. The sides of a triangular field are 115 yds., 181 yds., and 204 yds. Find the rent at £1 2*s.* per acre.

26. The rent of a triangular field at £1 10*s.* 3*d.* per acre is 12 guineas. If one of the sides be 240 yds., find the perpendicular on it from the opposite angle.

27. The sides of a triangular field are to each other as 13 : 14 : 15, and the cost of running wire netting round it at 4½*d.* per yd. is £4 14*s.* 6*d.* Find the area.

28. The sides of a triangular field are 60 yds. 1 ft., 86 yds. 1 ft., and 100 yds. If the rent is 10*s.* 9½*d.*, what is that per acre?

29. How many sq. yds. are there in a triangular field in which two adjacent sides are 480 yds. and 225 yds., and the included angle is 42° 34′? Given log 2 = .3010300, log 3 = .4771213, *L* sin 42° 34′ = 9.8302342, log 3.6528 = .5626259, log 3.6529 = .5626378.

30. Find in sq. yds. to two decimal places the area of a triangular enclosure, one side of which is 270 yds., the adjacent angles being 60° and 37° 20′, and reduce the answer to acres, &c. Given log 2 = .3010300, log 3 = .4771213, *L* sin 82° 40′ = 9.9964330, *L* sin 37° 20′ = 9.7827958, log 19301 = 4.2855798, *D* = 226.

CHAPTER II.—QUADRILATERALS

Sec. I.—Lengths

11. The following classification of four-sided figures will be found useful.

I. If *both* pairs of opposite sides are parallel, the figure is a *parallelogram.*

There are three particular cases of parallelograms.

(1) A *square* has all its angles right angles, and all its sides equal.

(2) A *rectangle* has all its angles right angles, but not all its sides equal.

(3) A *rhombus* has all its sides equal, but its angles are not right angles.

Other parallelograms, which are neither rectangular nor equilateral, are called by Euclid *rhomboids;* but the term is of no practical use, as the *class-name* parallelograms is always used instead.

II. If *one pair* of opposite sides only are parallel, the figure is a *trapezium* or *trapezoid.*

(1) A *trapezium* when both parallel sides are at right angles to a third side.

(2) A *trapezoid** when they are not at right angles to either of the other sides.

III. All other four-sided figures are simply called by the class-name *quadrilaterals.*

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* See note on page 28.

Practically quadrilaterals will be treated under the following heads, and in the following order: The rectangle—the square—the oblique-angled parallelogram—the rhombus—the trapezium—the trapezoid—other quadrilaterals.

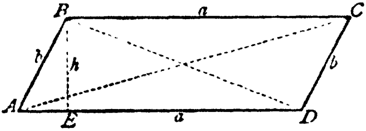
12. The *diagonal* of a quadrilateral is the straight line joining two opposite corners. Every quadrilateral has obviously two diagonals.

The *altitude* of a quadrilateral is the perpendicular drawn to the base from one of the opposite angular points. Any side may be taken for base, as in the triangle.

The problems to be solved in the present section are mainly connected with the determination of the *diagonal* and *altitude* in terms of the sides, and of one another. Since every quadrilateral is divided by its diagonal into two triangles, all these problems resolve themselves into problems on triangles; so that no new formulæ are required.

13. If a and b be the length and breadth of a *rectangle*, either diagonal will $=\sqrt{a^2+b^2}$. For the diagonal is the hypotenuse of a right-angled triangle, of which a and b are the sides. It follows that if a be the side of a square, either diagonal $=a\sqrt{2}$.

14. In a *parallelogram*, the angles A and B are supplementary.



$$BD^2=AB^2+AD^2-2AB\,.\,AD\cos A.$$

$$AC^2=AB^2+BC^2-2AB\,.\,BC\cos(180°-A).$$

$$=AB^2+BC^2+2AB\,.\,BC\cos A.$$

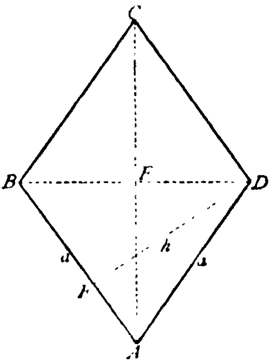
Thus the diagonals of a parallelogram are:

$$\sqrt{a^2+b^2-2ab\cos A} \text{ and } \sqrt{a^2+b^2+2ab\cos A}.$$

The altitude if a be taken for base $= b \sin A$.

,, if b ,, ,, $= a \sin A$.

15. The *rhombus* may be regarded as a particular case of the parallelogram, where $a = b$. As, however, the rhombus is an interesting figure, and scarcely treated of by Euclid, it will be worth while to examine its properties more in detail.



Its character will be best seen when it is situated as in the figure annexed, which reveals the well-known *diamond-shape.*

Euclid defines a rhombus as a four-sided figure which has all its sides equal, but its angles are not right angles.

We have already *assumed* that a rhombus is a parallelogram, and we can prove it readily from Euclid's definition.

For join AC, then the triangles BCA, DAC, have their sides equal, each to each; ∴ angle $BCA =$ angle CAD, and BC is parallel to AD. Similarly BA is also parallel to CD.

∴ $ABCD$ is a parallelogram, so that its opposite angles are equal, and its adjacent angles supplementary. If the adjacent angles also were equal, the rhombus would become a square; it is the essence of a rhombus that they should not be equal.

In the triangles BCA, DCA, the sides BC, $CA =$ the sides DC, CA each to each, and the bases BA, DA are equal. ∴ CA bisects the angle at C, also the angle at A. Similarly BD bisects the angles at B and D.

∴ *the diagonals of a rhombus bisect the angles.*

Again, if the diagonals intersect at E, in the triangles BCE, DCE the sides BC, CE = the sides DC, CE, and the angle BCE = angle DCE.

$\therefore$ base BE = base DE, and the angle CEB = the angle CED.

$\therefore$ each is a right angle.

Similarly BD bisects CA at right angles.

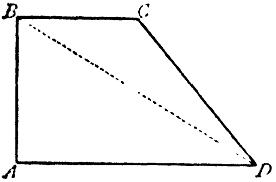
$\therefore$ *the diagonals of a rhombus bisect each other at right angles.*

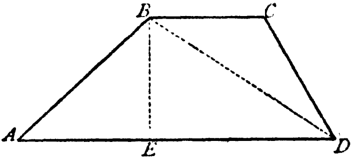
$$\left.\begin{aligned} BD &= 2BE = 2BA \sin BAE = 2a \sin \frac{A}{2}. \\ CA &= 2EA = 2BA \cos BAE = 2a \cos \frac{A}{2}. \end{aligned}\right\}$$

$\therefore$ the *diagonals* of a rhombus are $2a \sin \frac{A}{2}$, $2a \cos \frac{A}{2}$.

The altitude $= a \sin A$, whatever side be taken for base.

16. In the *trapezium*, if AB and AD are known, the





diagonal BD can be found as in the rectangle.

In the *trapezoid*, if AB, AD, and the angle A are known, the diagonal BD can be found as in the parallelogram.

In the trapezoid, the altitude is always the perpendicular distance between the two parallel sides. Thus altitude $BE = AB \sin A$ as before.

Examples.—(1) The diagonal of a square field is 250 yds. Find how many yards of wire netting will be required to surround it.

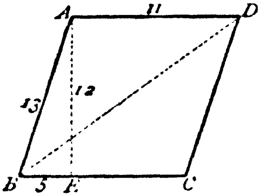
$$\text{Side of the field} = \frac{1}{\sqrt{2}} \times 250 \text{ yds.}$$

$$\therefore \text{ Perimeter} = \frac{4}{\sqrt{2}}\, 250 = 500\sqrt{2} \text{ yds.}$$

$$= 707 \cdot 1 \text{ yds.}$$

(Taking $\sqrt{2} = 1 \cdot 4142$.)

(2) The longer altitude of a parallelogram is 12 ft.; the sides are 13 ft. and 11 ft. Find the longer diagonal.



Since $AE = 12$ ft.,

$$BE = \sqrt{13^2 - 12^2} = 5 \text{ ft.}$$

Now

$$BD^2 = AB^2 + AD^2 + 2AB \,.\, AD \cos B.$$

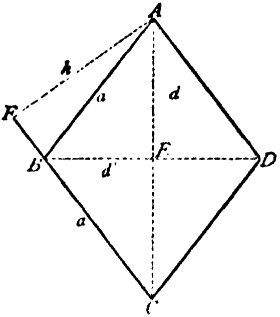
$$\text{and } \cos B = \frac{BE}{AB} = \frac{5}{13}.$$

$$\therefore\ BD^2 = 13^2 + 11^2 + 2 \,.\, 13 \,.\, 11 \,.\, \tfrac{5}{13} = 169 + 121 + 110.$$

$$= 400.$$

$\therefore\ BD = 20$ ft. Answer.

(3) If d, d' be the diagonals of a rhombus, find the side, and the altitude.



$$\text{Side } AB = \sqrt{AE^2 + BE^2}$$

$$= \tfrac{1}{2}\sqrt{d^2 + d'^2}.$$

From A draw AF perpendicular to FC, and denote it by h. Then, since the angle BAC = the angle BCA, the right-angled triangles AFC, ABE are similar.

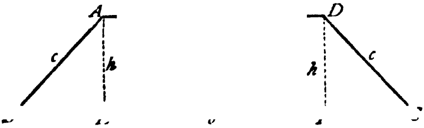
$$\therefore\ FA : AC :: EB : BA.$$

$$h : d :: \frac{d'}{2} : a.$$

$$\therefore\ h = \frac{dd'}{2a} = \frac{dd'}{\sqrt{d^2 + d'^2}}. \quad \text{Answer.}$$

(The problem can be solved without assuming the properties of similar triangles, but not so neatly.)

(4) A trapezoid has two sides parallel and two sides equal. If the parallel sides are a and b, b being the longer, and the other sides each $= c$, find the altitude.



Draw the perpendiculars AE, DF. Then, since $DF = AE$, and $AB = DC$, ∴ also $BE = FC$.

Now

Answer.

NOTE ON THE TERMS 'TRAPEZIUM' AND 'TRAPEZOID.'—Euclid, after defining other kinds of quadrilaterals, adds, "All other four-sided figures are called trapeziums." This practically means: All four-sided figures not parallelograms are called trapeziums. Some writers have followed Euclid; others have restricted 'trapezium' to the case in which two sides are parallel; while others have substituted the word 'trapezoid' for this case, and called quadrilaterals trapeziums when no two sides are parallel. I have ventured to introduce a further distinction. The word 'quadrilateral' is quite definite enough for the case in which no two sides are parallel; and in practice 'trapezium' is not used in this sense, but the class-name is substituted, just as in the case of the terms 'rhomboid' and 'scalene triangle' we practically say merely 'parallelogram' and 'triangle.' Thinking the word 'trapezium' too good to be lost, I have proposed the distinction drawn in the text, which I think is useful, and suggests itself.

EXAMPLES ON THE LENGTHS OF QUADRILATERALS

1. Find the diagonal of a square whose side is 2 fur. 10 po. 3 yds.

2. A path is made from corner to corner of a rectangular piece of ground 1 mile 220 yds. long, and 825 yds. broad. Find its length.

3. A road 2½ miles long runs straight from corner to corner of a square enclosure. Determine the cost of running a fence all round the enclosure, and also on each side of the road, at 5*d.* per foot, the breadth of the road not being taken into calculation.

4. Find the shorter diagonal of a parallelogram whose sides are 40 ft. and 35 ft., and included angle $33° 24'$. Given $\cos 33° 24' = .8348479$.

5. Find the diagonal of a rectangular room 16 ft. × 12 ft.

6. Given the diagonal of a square 1 mile 480 yds., find the side in yds., &c., correct to an inch.

7. Find the diagonal of a rectangle whose sides are 5 yds. 1 ft. and 10 yds.

8. The diagonal of a rectangle is 79 yds. 1 ft., and the sides are to each other as 8 : 15. Find the perimeter.

9. Find both the diagonals of a parallelogram whose sides are 4 ft. 8 in. and 5 ft. 4 in., and whose smaller angle is 60°.

10. The sides of a parallelogram are 8 ft. and 4 ft. Find both the altitudes, if the smaller angle is 15°.

11. The diagonals of a parallelogram are 3 in. and $\sqrt{73}$ in., and the shorter altitude is 3 in. Find the sides.

12. The two sides of a parallelogram are 10 in. and 1 ft. 9 in., and the shorter altitude is 8 in. Find the shorter diagonal.

13. What are the diagonals of a rhombus, one of whose angles is 30°, and whose side is $\sqrt{2}$?

14. In a rhombus, if the diagonals are 6 ft. and 8 ft., find the side and altitude.

15. If the side of a rhombus is a, and the altitude h, then the diagonals are $\sqrt{a(a+h)} \pm \sqrt{a(a-h)}$.

16. If the side of a rhombus be 4 in., and the altitude $2\sqrt{3}$ in., find the diagonals.

17. Also if the side be 13 in., and the altitude $9\frac{3}{13}$ in.

18. If the side of a rhombus be 1 ft. $2\frac{1}{2}$ in., and the longer diagonal be 1 ft. 9 in., find the shorter diagonal and altitude.

19. Given the altitude of a rhombus $=$ 1 ft. $2\frac{2}{17}$ in., and longer diagonal $=$ 2 ft. 6 in., find side and other diagonal.

20. Given the diagonals of a rhombus, show how to find the angles. If the diagonals are 9 in. and 4 in., find the angles. Given $\log 2 = .3010300$, $\log 3 = .4771213$, $L \tan 66° 2' = 10.3520972$, diff. for $1' = 3403$.

21. Given three sides of a trapezium, show how to find the fourth. If the two parallel sides are a and b, and the side at right angles to them is c, then the fourth side $= \sqrt{c^2 + (a-b)^2}$. Example: Let $a = 13$ ft., $b = 14$ ft. 7 in., $c = 15$ ft.

22. In a trapezoid, if the two parallel sides are a and b, and the third side c makes an obtuse angle θ with them, then the fourth side $= \sqrt{(a-b)^2 + 2(a-b)c \cos}$

23. Find both the diagonals of a trapezium whose parallel sides are 9 in. and 4 in., and whose longest remaining side is 13 in.

24. In a trapezoid, if the two non-parallel sides are equal (each $= a$), and are inclined at an angle θ to the third side (c), then the fourth side $= c - 2a \cos \theta$. Example: Let $c = 32$ yds., $a = 27$ yds., and $\theta = 60°$.

25. A trapezoid has two sides parallel and two sides equal; one of the equal sides is 25 ft., and the shorter of the parallel sides is 31 ft. Find the other parallel side, if the altitude is 24 ft.

26. A trapezoid has two sides parallel and two sides equal; the two parallel sides are 6 in. and 3 ft. 10 in., and the other sides are each 2 ft. 5 in. Find the altitude.

27. In a trapezoid, the two parallel sides are 11 in. and 2 ft. 4 in., and the other two sides are 2 ft. 1 in. and 2 ft. 2 in. Find the altitude, and also the shortest diagonal.

28. Two opposite angles of a quadrilateral are right angles. The sides adjacent to one of them are a and b, and a third side is c. Find the fourth side. Example: Let $a = 1$ ft. 4 in., $b = 5$ ft. 3 in., $c = 2$ ft. 1 in.

29. In a quadrilateral, given three sides and two included angles, show how to find the fourth side. Given the sides 64 ft., 128 ft., and $64\sqrt{2}$ ft., and the included angles 60° and 45°, find the fourth side.

30. If a, b, c, d are the sides of a quadrilateral given in order, and the angles between the first two and the last two sides are supplementary, show that the diagonal joining the two other angular points is $\sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$. If the sides are 9, 10, 11, and 12 feet, find this diagonal correct to an inch.

Section II.

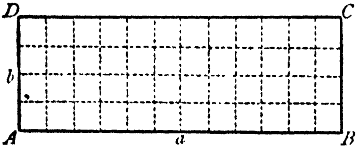
[FORMULÆ.

(1) Area of *rectangle* = length × breadth.
(2) „ *square* = $(\text{side})^2$.
(3) „ *parallelogram* = base × altitude.
(4) „ *rhombus* = ½ product of diagonals.
(5) „ *trapezium* = ½ sum of parallel sides × third side.
(6) „ *trapezoid* = ½ sum of parallel sides × altitude.]

17. *The rectangle* (area = ab).

Suppose the unit of linear measure to be 1 inch, then the unit of square measure will be 1 square inch; i.e. a square which is 1 inch both in length and breadth.

Let the length AB of the rectangle be a inches, and the breadth AD b inches. Divide AB into a equal parts, and AD into b equal parts; each of these parts will be one inch. Draw through these points parallel straight lines, as in the figure, dividing the rectangle into parts, each of which is exactly one square inch. Now the lines drawn across the rectangle divide it into b rows, and each row contains a square inches. Therefore altogether in the figure there are $a \times b$ square inches.



∴ area of rectangle = ab; i.e. length × breadth. Q.E.D.

The reasoning in the above proof should be carefully observed, for it is the foundation of all theorems regarding the areas of plane figures. For instance, the area of a triangle depends on it, and consequently it was necessary to assume it in the last chapter.

It is obvious that the character of the reasoning is not altered, whatever unit be taken. Thus, if the length were 8 yds., and the breadth 6 yds., the area would be $8 \times 6 = 48$ sq. yards. Or, if the length were $\frac{3}{8}$ of a mile, and the breadth $\frac{1}{3}$ of a mile, the area would be $\frac{3}{8} \times \frac{1}{3} = \frac{1}{8}$ of a sq. mile.

Examples.—The rectangle is a figure which meets us very often in practical questions, such as carpeting floors, papering walls, paving courtyards, &c.

(1) Find the expense of carpeting, at 4*s.* 6*d.* per sq. yd., a room 18 ft. long, and 12 ft. 9 in. broad, leaving a margin of 18 inches all round for staining.

Since 18 in. has to be taken away on both sides, length of carpet = 18 ft. − 3 ft. = 15 ft., and breadth = 12 ft. 9 in. − 3 ft. = $9\frac{3}{4}$ ft.

$$\therefore \text{ area of carpet} = 15 \times 9\tfrac{3}{4} \text{ sq. ft.} = \frac{1}{9} \times 15 \times \frac{39}{4} \text{ sq. yds.}$$

$$\text{Cost of carpet} = 4\tfrac{1}{2} \times \frac{1}{9} \times 15 \times \frac{39}{4}\, s.$$

$$= \frac{9}{2} \times \frac{1}{9} \times \frac{15 \times 39}{4} = 73\tfrac{1}{8}\, s.$$

$$= £3\ 13s.\ 1\tfrac{1}{2}d. \quad \text{Answer.}$$

(2) Find the cost of papering the walls of a room 19 ft. 5 in. long, 12 ft. 7 in. broad, and 8 ft. 3 in. high, with paper 27 in. wide, and costing 3*d.* per yard.

(If the walls of a room were placed side by side in a straight line, they would form a rectangle of which the *height* of the room is the breadth, and the *perimeter* of the room the length. Hence we obtain the most convenient formula for practice.

Area of walls = perimeter × height.

The perimeter is obviously twice the length added to twice the breadth.)

$$\begin{array}{lrr} & \text{ft.} & \text{in.} \\ \text{Here length} = & 19 & 5 \\ \text{breadth} & 12 & 7 \\ \hline & 32 & 0 \\ & & 2 \\ \hline \text{perimeter} = & 64 & 0 \end{array}$$

$$\therefore \text{ area of walls} = 8\tfrac{1}{4} \times 64 = \frac{33}{4} \times \overset{16}{64} \text{ sq. ft.}$$

$$= \frac{\overset{11}{33} \times 16}{\underset{3}{9}} = \frac{176}{3} \text{ sq. yds.}$$

When the paper is of a given width, we must divide the area of the walls by it, and the result will be the number of linear yards (or feet) of paper.

$$\text{Here width of paper} = 27 \text{ in.} = \tfrac{3}{4} \text{ yd.}$$

$$\therefore \text{ No. of yds. of paper} = \tfrac{176}{3} \times \tfrac{4}{3}.$$

$$\text{Cost of paper} = \frac{176}{3} \times \frac{4}{3} \times 3 = \frac{704}{3} d.$$

$$= 19s.\ 6\tfrac{2}{3}d. \quad \text{Answer.}$$

(In practical examples doors and windows have to be allowed for, their area being subtracted from the whole area of the walls.)

(3) How many tiles, each 9 × 6 in., will be required to pave a courtyard 30 ft. × 25 ft.?

$$\text{Area of courtyard} = 30 \times 25 \text{ sq. ft.} = 30 \times 25 \times 144 \text{ sq. in.}$$

$$\text{Area of each tile} = 9 \times 6 \text{ sq. in.}$$

$$\therefore \text{ number of tiles} = \frac{30 \times 25 \times 144}{9 \times 6} = 2000 \text{ tiles.}$$

(4) *Duodecimal Multiplication.*—The labour of multiplying length by breadth, where both are expressed in feet and inches, is often shortened as in the following example.

Find the area of a floor 12 ft. 7 in. × 9 ft. 10 in.

$$\begin{array}{rrr}
\text{ft.} & \text{in} & \\
12 & 7 & \\
9 & 10 & \\
\hline
113 & 3 & \\
10 & 5 & 10 \\
\hline
123 & 8 & 10 \\
\hline
\end{array}$$

RULE.—Place the two dimensions under each other. Multiply the top line by the number of feet; then multiply by the number of inches, beginning one place nearer the right; then add the two lines together. *In every case carry at 12.*

The first number in the result is got by multiplying feet by feet, and therefore represents square feet.

The second is got by multiplying feet by inches; therefore every unit here represents 12 square inches. These are usually called *primes* (or superficial primes).

The third is got by multiplying inches by inches, and therefore represents square inches.

Thus the above result = 123 sq. ft. 8 *primes* 10 sq. in. = 123 sq. ft. 8 × 12 + 10 sq. in. = 123 sq. ft. 106 sq. in. Answer.

18. *The square* (area = a^2).

The square is a particular case of the rectangle, where the length and breadth are equal, so that $b = a$. Hence area = a^2; or, to adopt the explanation of the last paragraph, if the side contain a inches, there will be a rows, each containing a square inches; so that there are a^2 square inches in the whole figure.

Examples.—(1) Find the area of a square field, the side of which is 8 chains 25 links.

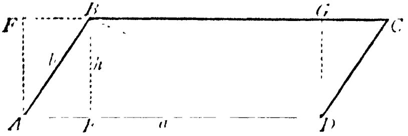
```
8 chains 25 links =      8·25 chains.
                         8·25
                       ------
                         4125
                        1650
                       6600
                       ------
                   10 ) 68·0625 sq. chains.

                       6·80625 acres.
                             4
                       -------
                       3·22500
                         40
                       -------
                       9·000        Answer 6 a. 3 r. 9 p.
```

(2) Find the cost of running a fence round a square piece of ground 10 acres in extent at 5*d.* per yd.

10 acres = $\frac{1}{64}$ mile.
∴ side of square = $\frac{1}{8}$ mile.
perimeter „ = $\frac{1}{2}$ mile = 880 yards.
∴ cost = 880 × 5*d.* = 4400*d.*
= £18 6*s.* 8*d.* **Answer.**

19. *The parallelogram (oblique).*



$$\left.\begin{aligned}\text{Area} &= ah.\\ &= ab \sin A.\end{aligned}\right\}$$

Let BE be the altitude of the parallelogram $ABCD$. Draw AF, DG parallel to BE to meet BC.

$$\begin{aligned}\therefore\ \text{area of } ABCD &= \text{rectangle } FADG. \quad \text{(Euclid, i. 35.)}\\ &= AD\,.\,AF.\\ &= AD\,.\,BE.\\ &= ah \quad \ldots\ldots \quad (1).\end{aligned}$$

Since $BE = AB \sin BAD$,

$$\text{area} = ab \sin A \quad \ldots\ldots \quad (2).$$

We might deduce these results from the expressions for the area of a triangle obtained in the last chapter. For area of the triangle $BAD = \frac{ah}{2} = \frac{1}{2} ab \sin A$; and the parallelogram $ABCD$ is double of the triangle BAD, since it is bisected by the diagonal.

It seems best, however, to deduce the area of the parallelogram from that of the rectangle, on which both proofs ultimately depend.

Example.—The sides of a parallelogram are as 3 : 4, and the included angle is 30°. Find the sides, if the area = 1734 sq. yds.

Let $3x$ and $4x$ be the sides.

$$\therefore\ 3x \times 4x \sin 30° = 1734.$$

$$\frac{3x \times 4x}{2} = 1734.$$

$$x^2 = 289.$$

$$x = 17.$$

∴ the sides are 51 yds. and 68 yds.

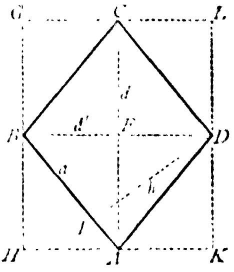
20. *The rhombus* (area $= ah = a^2 \sin A = \frac{dd'}{2}$).

Since the rhombus is a parallelogram, its area $= ah$ . . . (1).

But $h = a \sin A$.

$\therefore$ area $= a^2 \sin A$ . . . (2).

But there is another far more convenient form in which the area can be expressed.



The diagonal AC divides the rhombus into two triangles, ACB, ACD.

Now area of $ACB = \frac{1}{2} BE \cdot CA$.

" $ACD = \frac{1}{2} ED \cdot CA$.

$\therefore$ area of rhombus $= \frac{1}{2} (BE + ED)\, CA$.

$= \frac{1}{2} BD \cdot CA$.

$= \frac{1}{2}$ product of diagonals . . . (3). Q.E.D.

If the diagonals are d, d', then area $= \frac{dd'}{2}$.

The same result could be obtained by drawing parallels to the diagonals through the angular points A, B, C, D. For the rectangle $GHKL$ is obviously double the rhombus, since the sides of the rhombus bisect the rectangles GE, EK, HE, EL respectively.

$\therefore$ area $= \frac{1}{2}$ rectangle $GK = \frac{1}{2} AC \times BD$ (as before).

It is clear that this method of proof will apply whenever the diagonals are at right angles.

$\therefore$ generally, if the diagonals of a quadrilateral intersect at right angles, the area = half their product.

Examples.—(1) The diagonals of a hatchment are 9 ft. 8 in. and 7 ft. Find the cost of emblazoning it at £2 14*s.* per sq. yd.

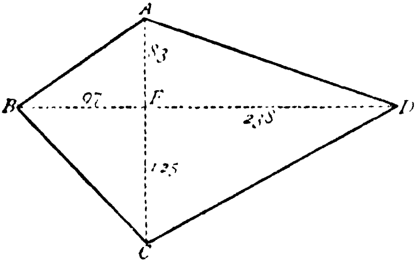
$$\text{Area of hatchment} = \tfrac{1}{2} \times 9\tfrac{2}{3} \times 7 \text{ sq. ft.}$$
$$= \tfrac{1}{2} \times \tfrac{29}{3} \times \tfrac{7}{9} \text{ sq. yds.}$$
$$\text{Cost} = 54 \times \frac{1}{2} \times \frac{29}{3} \times \frac{7}{9} \ s.$$
$$= 203s. = £10\ 3s. \quad \text{Answer.}$$

(2) The side of a rhombus is 2 ft. 5 in., and the shorter diagonal is 3 ft. 4 in. Find the area.

A reference to the last figure will show that $\dfrac{d^2 + d'^2}{4} = a^2$.

$$\therefore\ d'^2 = 4a^2 - d^2 = 4 \times 29^2 - 40^2.$$
$$= 18 \times 98 = 36 \times 49.$$
$$\therefore\ d' = 6 \times 7 = 42 \text{ in.}$$
$$\text{Area} = \frac{dd'}{2} = \frac{42 \times 40}{2} = 20 \times 42 \text{ sq. in.}$$
$$= 840 \text{ sq. in.} = 5 \text{ sq. ft. } 120 \text{ sq. in.} \quad \text{Answer.}$$

(3) *ABCD* is a field such that the straight lines joining opposite corners, *A*, *C*, and *B*, *D*, meet at right angles at *F*; *FA*, *FB*, *FC*, *FD* measuring 83, 97, 125, and 238 yards respectively. Find the area of the field in acres, roods, &c. (*Sandhurst.*)



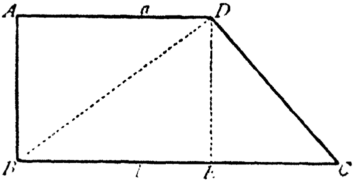
Since the diagonals are at right angles, the area is found by the same rules as in the rhombus.

$$\text{Diagonal } AC = 83 + 125 = 208 \text{ yds.}$$
$$\text{,,} \quad BD = 97 + 238 = 335 \text{ yds.}$$
$$\therefore\ \text{area} = \tfrac{1}{2} \times 208 \times 335 = 104 \times \overset{\text{sq. yds.}}{335} = \overset{\text{sq. yds.}}{34840}.$$

Reduced to acres, roods, &c., the area becomes:

7 a. 31 p. 22½ sq. yds. Answer.

21. (i) *The trapezium* (area $= \dfrac{a+b}{2} \,.\, c$).



Let a, b be the two parallel sides, and c the third to which they are at right angles.

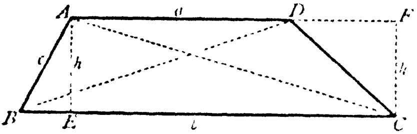
Join BD, and draw the perpendicular DE.

$\therefore$ area of triangle $BDC = \frac{1}{2}\, DE \,.\, BC = \frac{1}{2}\, bc.$

" $ABD = \frac{1}{2}\, AB \,.\, AD = \frac{1}{2}\, ac.$

$\therefore$ area of trapezium $= \frac{1}{2}\,(a+b) \,.\, c.$

(ii) *The trapezoid* (area $= \dfrac{a+b}{2} \,.\, h$).



Let a, b be the two parallel sides, and h the altitude.

Join AC, and draw the perpendiculars AE, CF, both of which will $= h$.

$\therefore$ area of triangle $ACD = \frac{1}{2} \,.\, AD \,.\, CF = \frac{1}{2}\, ah.$

" $ABC = \frac{1}{2} \,.\, BC \,.\, AE = \frac{1}{2}\, bh.$

$\therefore$ area of trapezoid $= \frac{1}{2}\,(a+b)\, h$ . . . . (1).

If the angle A is known, then $h = AB \sin B$.

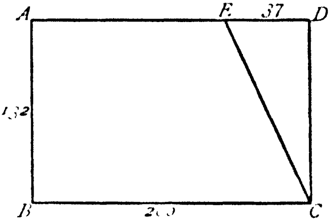
$= AB \sin A = c \sin A.$

$\therefore$ area $= \frac{1}{2}\,(a+b)\, c \sin A$ . . . . . (2).

The trapezoid may be considered to bear the same relation to the trapezium that the parallelogram (oblique) bears to the rectangle.

Examples.—(1) A rectangular field is 200 yds. long, and 132 yds. broad. A road is made, passing through a corner, and cutting off 37 yds. from the length. Find the area of the remaining part.

We have to find the area of the trapezium $ABCE$.



$$\text{Area} = \tfrac{1}{2}(AE + BC) \,.\, AB,$$

and $$AE = AD - DE = 200 - 37 = 163 \text{ yds.}$$

$$\therefore \text{ area} = \tfrac{1}{2}(163 + 200) \,.\, 132 = 363 \times 66 \text{ sq. yds.}$$

$$= \frac{\overset{3}{363} \times 66 \times 4}{121} \text{ sq. poles} = 792 \text{ sq. poles.}$$

$$= 4 \text{ a. } 3 \text{ r. } 32 \text{ p.} \quad \text{Answer.}$$

(2) A field is in shape a trapezoid, whose parallel sides are 6 chains 75 links, and 9 chains 25 links. If the area be 2 a. 3 r. 8 p., find the shortest way across the field in yards.

The shortest way will be the perpendicular distance between the parallel sides; i.e. the *altitude* of the trapezoid.

$$\text{Now area of trapezoid} = \tfrac{1}{2}\overset{\text{chains.}}{(6{\cdot}75 + 9{\cdot}25)} \times \text{altitude.}$$

$$\therefore 8 \text{ chains} \times \text{altitude} = 2 \text{ a. } 3 \text{ r. } 8 \text{ p.}$$

∴ if we reduce the area to sq. chains, and divide by 8, the result will be the altitude in chains.

40) 8·0

4) 3 2

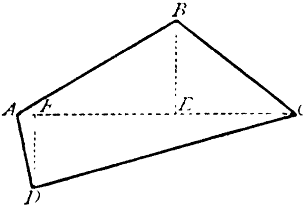
2·8 acres.
10

8) 28 sq. chains.

3·5 = altitude in chains.
22

77 yds. Answer 77 yds.

22. In *any quadrilateral* we can readily determine the area by dividing it into two triangles.

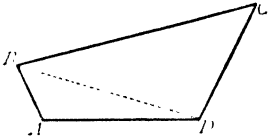


For instance, if the diagonal AC is given, and the perpendiculars on it BE, DF, then

$$\text{Area of quadrilateral} = \tfrac{1}{2}\, BE \,.\, AC + \tfrac{1}{2}\, FD \,.\, AC.$$
$$= \tfrac{1}{2}\,(BE + FD)\,.\,AC.$$

Hence, if only the diagonal and *the sum* of the perpendiculars on the diagonal be given, we can determine the area.

For another instance, suppose the four sides and one angle A are given.



Draw the diagonal BD, the value of which can be determined. Then the quadrilateral is the sum of two triangles, in one of which we know two sides and the included angle, and in the other the three sides. Hence its area can be determined.

Examples.—(1) The longer diagonal of a quadrilateral is 146 yds., and the sum of the perpendiculars on it from the other two angles is 93 yds. Find the area in acres, &c.

Area $= \frac{1}{2} \,.\, 146 \times 93$ sq. yds. $= 73 \times 93$ sq. yds. $= 6789$ sq. yds.

This reduces to 1 a. 1 r. 24 p. 13 sq. yds. Answer.

(2) The sides of a quadrilateral $ABCD$, with a right angle at A, are $AB = 3$ ft., $BC = 4$ ft. 8 in., $CD = 4$ ft. 4 in., $DA = 4$ ft. Find the area.

$$BD = \sqrt{3^2 + 4^2} = 5 \text{ ft.}$$

Area of triangle BCD, in which we know the sides

$$= \sqrt{7\,(7-5)\,(7-4\tfrac{2}{3})\,(7-4\tfrac{1}{3})} = \sqrt{7 \,.\, 2 \,.\, \tfrac{7}{3} \,.\, \tfrac{8}{3}} \text{ sq. ft.}$$
$$= \tfrac{28}{3} \text{ sq. ft.} = 9\tfrac{1}{3} \text{ sq. ft.}$$

And area of right-angled triangle $BAD = \dfrac{3 \,.\, 4}{2} = 6$ sq. ft.

$\therefore$ area of quadrilateral $= 15\tfrac{1}{3}$ sq. ft.

EXAMPLES ON THE AREA OF QUADRILATERALS

(a) Square

1. Find the area of a square field whose side is a quarter of a mile.

2. A road, 1 mile 6 fur. long, leads straight from corner to corner of a square enclosed common. Find the area of the common.

3. The perimeter of a square field is 7 furlongs. Find its area.

4. Find the side of a square field containing 19 a. 2 r. 16 p.

5. If the cost of surrounding a square field with wire netting, at 5½*d.* per ft., was £39 6*s.* 6*d.*, find the area.

6. Cost of running a fence round a square field of 40 acres, at 5½*d.* per yd.

7. Cost of carpeting a square floor, whose side is 18 ft. 4 in., with linoleum at 2*s.* 9*d.* per sq. yd.

8. Cost of levelling a path 4 ft. wide to surround a square piece of turf containing 784 sq. yds., at 3*s.* for 8 sq. yds.

9. The outside part of a field containing 4 a. 2 r. 23 p. 24¼ sq. yds. is taken up by a path 1½ yds. wide. By how much will the path lessen the area of the field?

10. How many chains are there in the side of a square field of 9 a. 2 r. 1 p.?

(b) Rectangle

11. Find the area in sq. yds. of a rectangular floor 12 ft. 9 in. by 21 ft. 4 in.

12. Find the area of a rectangular field $\frac{1}{10}$ of a mile in length, and 147 yds. 1 ft. in breadth.

13. Find the side of a square equal in area to a rectangle whose sides are 50 ft. 5 in. and 33 ft. 9 in.

14. In a rectangular field the area is 4 a. 0 r. 8 p., and one side is double the other. Find the sides.

15. Find the number of bricks 13 in. by 6 in. in a rectangular wall 1521 ft. long and 8 ft. high.

16. Required the breadth of a rectangle whose length is four times the breadth, and whose area is 746,496 sq. yds

17. The frame of a picture is 3 in. broad, and its outside dimensions are 18 in. by 10 in. Find area of picture and frame.

18. A room is 17 ft. 6 in. long, 16 ft. 3 in. broad, and 10 ft. 8 in. high. Find the cost of papering the walls with paper 27 in. wide, which cost 4½*d.* per yd.

19. Find the cost of painting the walls of a room 19 ft. 10¼ in. by 16 ft. 1¾ in., and 10 ft. 3 in. high, at 9½*d.* the sq. yd.

20. Find the cost of carpeting a room 18 ft. 9 in. long, by 12 ft. 6 in. broad with carpet ¾ yd. wide at 3*s.* 9*d.* per yd.

21. Find the expense of whitewashing the walls and ceiling of a room 17 ft. 4 in. long, 13 ft. 6 in. broad, and 10 ft. 9½ in. high, at 3½*d.* per sq. yd.

22. If the expense of carpeting a room with carpet 27 in. wide, at 4*s.* 6*d.* per yd., is £15 6*s* 8*d.*, and the expense of papering the walls with paper 18 in wide, at 5¼*d.* per yd., is £4 19*s.* 10¾*d.*, and the breadth is $\frac{15}{22}$ of the length, find the height.

23. The length and breadth of a room are 27 ft. 9 in. and 20 ft. 3 in., and the height is 9 ft 4 in. Find the cost of papering the walls with paper 27 in. wide at 6¼*d.* per yd., allowing for two windows, each 4 ft. 2 in. by 3 ft. 6 in., and a door 6 ft. 3 in. by 4 ft. 4 in.

24. A carpet 9 ft. by 7½ ft. is placed in the centre of a room, and surrounded by matting ½ yd. wide at 1*s.* 6*d.* per yd. The carpet being 3*s.* per yd., find the cost of carpet and matting together.

25. The area of the floor of a square room is 25 sq. yds., and the height of the walls is 10 ft. Find the cost of papering the walls with paper 27 in. wide at 2*s* 6*d.* per piece of 12 yds.

26. The cost of painting the walls of a room at 6½*d.* per sq. ft. amounts to £21 12*s.* 9½*d.* and the perimeter is 23 yds. 1 ft. 6 in. Find the height.

27. In a room 22 ft. long by 18 ft. wide there is a carpet 16 ft. square. Find the cost of covering the rest of the room with felt 2 ft. wide at 5*s.* per yd.

28. Find the cost of papering a room 17 ft. 7 in. long by 11 ft. 5 in. wide, and 10 ft. high, with paper 2 ft. 8 in. wide at 8*d.* per yd. (*Sandhurst.*)

29. The sum of £9 0*s.* 10*d.* is allowed for papering a room 27.7 ft. long, 19.55 ft. wide, and 12.4 ft. high. How much per yd. must be given for a paper 2.7 ft. wide? (*Sandhurst.*)

30. The length of a hall is three times the breadth; the cost of whitewashing the ceiling at 5⅓*d.* per sq. yd. is £4 12*s.* 7·1*d.*, and the cost of papering the four walls at 1*s.* 9*d.* per sq. yd. is £35. Find the height of the hall. (*Sandhurst.*)

31. The walls of a library 32 ft. 8 in. long, 20 ft. 5 in. broad, and 17 ft. 5 in. high, are divided by bookshelves into compartments 4 ft. 1 in. long, and 1 ft. 7 in. high. If each compartment contains 21 books, valued at 2*s.* each, what is the value of the library?

32. Find the number of paving-stones 2 ft. square required to pave both sides of a street ⅜ mile long with a pavement 6 ft. broad. Find also the cost at 3*s.* 6*d.* per sq. yd.

33. A quadrangular court is 104 ft. by 86 ft.; a pathway 4 ft. wide runs round it, and the rest is turfed. Find the area of the turf.

34. Find the rent, at £2 6*s.* 8*d.* per acre, of a rectangular field 1 fur. 5 po. long by 1 fur. 23 po. broad.

35. If the rent of a rectangular field, at £1 13*s.* per acre, be £2 1*s.* 3*d.*, and the cost of running a fence round it, at 8*d.* per yd., is £11, find the length and breadth.

36. A rectangular field is 60 yds. long by 40 yds. wide; it is surrounded by a road of uniform width, the whole area of which is equal to the area of the field. Find the width of the road. (*Sandhurst.*)

37. Find the rent, at £2 5*s.* an acre, of a rectangular park ½ mile long, and ¼ mile wide. (*Sandhurst.*)

38. A rectangular garden contains 1200 sq. yds, and the length is to the breadth as 4 to 3. What will the fencing cost at 3*s.* 6*d.* the yd.? (*Sandhurst.*)

39. A rectangular court is 20 yds. longer than it is broad, and its area is 4524 sq. yds. Find its length and breadth. (*Sandhurst.*)

40. If 3 yds. be taken from one side of a rectangle whose perimeter is 14 yds., and added to the other side, its area will be doubled. Find the lengths of the sides. (*Sandhurst.*)

(c) Parallelogram (Oblique-angled)

41. Given the base of a parallelogram 23 yds. 1 ft. 9 in., and the altitude 14 yds. 2 ft. 5 in., find the area.

42. If the two sides of a parallelogram are 3 ft. 2 in. and 1 ft. 5 in., and the included angle is 60, find the area correct to nearest sq. in.

43. If the area of a parallelogram is 36 sq. yds. 5 sq. ft. 132 sq. in., and the altitude is 9 ft. 3 in., find the base.

44. If the area of a parallelogram is 95 sq. yds. 1 ft. 96 in., and the base is 28 yds. 1 ft. 8 in., find the altitude.

45. *ABCD* is a parallelogram 3 a. 2 r. 25 p. in area. If the perpendiculars from *C* on *AB* and *AD* are 82½ yds. and 49½ yds. respectively, find the two sides.

46. In a parallelogram whose area is 5 sq. ft. 108 sq. in., find the two sides, if the perpendicular on the base is 9 in., and the smaller angle is 30°.

47. The sides of a parallelogram are in the ratio 3 : 2, and the included angle is 45°. If the area is 264 sq. in., find the two sides correct to $\frac{1}{100}$ of a sq. in.

48. A field is in shape a parallelogram, two adjacent sides being 147 yds. and 343 yds., and the included angle being 15° 33′. Find the area in acres and sq. yds. Given log 3 = .4771213, log 7 = .8450980, *L* sin 15° 33′ = 9.4282631, log 13516 = 4.1308482, *D* = 321.

(d) Rhombus

49. Find the area of a rhombus whose diagonals are 3 ft. 4 in. and 5 ft. 3 in.

50. Find the area of a rhombus whose side = 2 yds., and one of whose angles is 75°.

51. Find the area of a rhombus whose side is 110 yds., and whose altitude is 77 yds.

52. The area of a rhombus is 4 a. 1 r. 20 p., and one of the diagonals is 385 yds. Find the other.

53. Each side of a rhombus is 120 yds., and two of its opposite angles are each 60°. Find the area of the rhombus in acres to two decimal places. (*Sandhurst.*)

54. A rectangular lawn 60 ft. by 40 ft. has four rectangular beds 20 ft. by 6 ft. cut out of it, and a diamond-shaped bed in the centre, whose diagonals are 15 ft. and 8 ft. Find the area of grass remaining.

55. Five diamond-shaped beds are cut out of a rectangular lawn, in the shape of the five of diamonds. If *ACBD* be the central diamond, and its diagonals *BOA*, *DOC* produced meet the edge of the lawn at *E* and *F* respectively, find what proportion of the lawn is occupied by the five beds. Given $AO = \frac{1}{4} AE$, and $OC = \frac{1}{3} CF$.

56. A playing-card (the ten of diamonds) is $3\frac{5}{8}$ in. by $2\frac{1}{2}$ in. If the diagonals of every diamond on the card are $\frac{3}{8}$ in. and $\frac{9}{16}$ in., find how much of the surface of the card is uncoloured.

57. The surface of a playing-card contains $9\frac{1}{16}$ sq. in., and the diagonals of the diamonds on it are $\frac{3}{8}$ in. and $\frac{9}{16}$ in. If the diamonds take up $\frac{9}{38}$ of the whole surface, how many diamonds are there on the card?

58. A lattice window with diamond panes is 4 ft. 6 in. by 2 ft., and has eleven bars crossing from right to left, and eleven from left to right, all at equal distances and the same angle. Find the number of panes (counting in half-panes at the side) into which the window is divided, and the area of each.

(e) Trapezium and Trapezoid

59. Find the area of a trapezium, if the lengths of the two parallel sides are 54 yds. and 67 yds., and the length of the side at right angles to them is 40 yds.

60. Find the area of a trapezoid, if the lengths of the two parallel sides are 7 ft. 5 in. and 8 ft. 7 in., and the perpendicular distance between them is 6 ft. 3 in.

61. In the trapezoid $ABCD$ the sides AB, DC are parallel. Given the length of BC (18 in.), and the perpendiculars on it from A and D (9 in. and 14 in.), find the area.

62. A field is in the form of a trapezoid. Its parallel sides are respectively 10 chains 30 links and 7 chains 70 links; the distance between them is 7 chains 50 links. Find the acreage. (*Sandhurst.*)

63. $ABCD$ is a quadrilateral field, in which the angles C, D are right angles, and the angle A is half a right angle. Find the area of the field in acres, &c., having given that $BC = 91$ yds., and $AD = 151$ yds. (*Sandhurst.*)

64. Two of the sides of a trapezoid are parallel, and two are equal. If the parallel sides are a and b, and the equal sides each $= c$, show that the area $= \dfrac{a+b}{4} \cdot \sqrt{4c^2 - (a-b)^2}$.

The front side of the roof of a house is in shape a trapezoid with two equal sides. The top edge is 14 ft., the bottom edge 24 ft., and each of the side edges 13 ft. Find the area of the whole side.

65. If £4 4*s.* 6*d.* be paid in rent for a field, in shape a trapezoid, the sum of whose parallel sides is 8 chains 45 links, and whose altitude is 6 chains, find the rent per acre.

66. The rent of a field, in shape a trapezoid, is £6 14*s.* 2*d.* at £2 0*s.* 4*d.* per acre, and the sum of the two parallel sides is 230 yds. Find the shortest distance across the field.

67. A straight belt of wood, 200 acres in extent, has two parallel sides, one of 3 miles in length, the other 3¼ miles; the other two sides completing the trapezoid. Find the length of the shortest path which can be cut through the wood.

68. In a trapezoid, if the two parallel sides are 3 in. and 5 in., and the third side (4 in.) be inclined at an angle 26° to the parallel sides, find the area. (Given $\log 2 = .3010300$, $L \sin 26° = 9.6418420$, $\log 70139 = 4.8459596$, $D = 62$.

(f) Quadrilateral (General)

69. In a quadrilateral field one diagonal is 8 chains, and the perpendiculars on it from the other angular points are 2 chains 50 links and 1 chain 75 links. Find the area.

70. In a quadrilateral the sum of the perpendiculars on a diagonal 3 ft. 6 in. in length is 2 ft. 1 in. Find the length of the other diagonal, the sum of the perpendiculars on it being 2 ft. 6 in.

71. Find the area of a quadrilateral field $ABCD$, in which the angle at A is a right angle, and the sides are $AB = 99$ yds., $BC = 143$ yds., $CD = 154$ yds., $DA = 132$ yds.

72. The shorter diagonal BD of a quadrilateral field $ABCD$ is 72 yds.; the sides adjacent to A are 127 yds. 1 ft. 6 in. and 136 yds. 1 ft. 6 in.; and the perpendicular from C on BD is 75 yds. 2 ft. Find the area.

73. Show that the area of any quadrilateral is half the parallelogram formed by lines equal and parallel to its diagonals.

The diagonals of a quadrilateral are 10 in. and $10\sqrt{3}$ in., and meet at an angle of 60°. Find the area.

74. The sides of a quadrilateral $ABCD$ are $AB = 1$ ft. 8 in., $BC = 2$ ft. 10 in., $CD = 3$ ft. 3 in., $DA = 3$ ft. 9 in.; the diagonal $AC = 3$ ft. 6 in. Find the area.

75. If two opposite angles A and C of a quadrilateral $ABCD$ are right angles, and $AB = 5$ ft. 4 in., $AD = 21$ ft., $BC = 8$ ft. 4 in., find the area.

76. AB is the diameter of a circle, C and D two points on the circumference on opposite sides. If $AB = 25$ ft., $AC = 7$ ft., $BD = 15$ ft., find the area of quadrilateral $ACBD$.

77. A quadrilateral field $ABCD$ has its sides as follows: $AB = 3$ chains, $BC = 1$ chain, $CD = 2$ chains, $DA = 4$ chains, and the angle $ABC = 120°$. Find the area correct to a sq. pole. ($\sqrt{3} = 4.796$.)

78. On opposite sides of a base, which is 120 yds. long, two isosceles triangles are constructed. The altitude of one triangle is double the altitude of the other, and the triangle that has the least altitude has a right angle for its angle opposite the base. Find in sq. yds. the area of the four-sided figure thus formed, and express the result also in acres, roods, &c. (*Sandhurst.*)

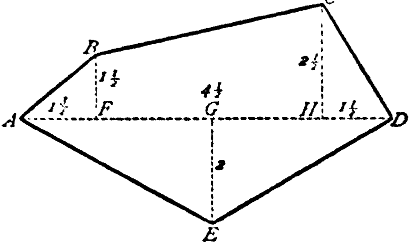
CHAPTER III.—IRREGULAR POLYGONS

23. A polygon is called *regular* when it has all its sides equal, and all its angles equal. When it does not comply with both these conditions it is called *irregular.*

24. A straight line joining two angular points not adjacent is called a *diagonal.* No special rules need be laid down for the determination of diagonals beyond what has been said in the last chapter.

25. Any irregular polygon can be divided, by drawing diagonals, into triangles, whose number is always two less than the number of sides in the polygon. The areas of these triangles can be found if sufficient data be given, and the area of the polygon is their sum. Sometimes it will be found more convenient to divide them into parts, some of which are rectangles or trapeziums. An example or two will sufficiently illustrate the various cases.

Examples.—(1) *ABCDE* is a pentagonal field, of which *AFGHD* is the longest diagonal, *F*, *G*, *H* being the feet of the perpendiculars *BF*, *EG*, *CH*. Given that *AF* = 1 chain 75 links, *FH* = 4 chains 50 links, *HD* = 1 chain 50 links; and also *BF* = 1 chain 50 links, *EG* = 2 chains, *CH* = 2 chains 50 links, find the area of the field.



Here the pentagon is divided into four measurable parts: the right-angled triangles *ABF*, *CHD*, the trapezium *BH*, and the triangle *AED*.

sq. chains.

Area of right-angled triangle $AFB = \frac{1}{2}(1\cdot50 \times 1\cdot75)$.
$= \cdot8125$ sq. chains.

" " $CHD = \frac{1}{2}(2\cdot50 \times 1\cdot50)$.
$= 1\cdot875$ sq. chains.

Area of trapezium $BH = \frac{1}{2}(1\cdot50 + 2\cdot50) \times 4\cdot50$.
$= 9$ sq. chains.

" triangle $AED = \frac{1}{2}(2 \times 7\cdot75.)$
$= 7\cdot75$ sq. chains.

∴ area of pentagon $= 19\cdot4375$ sq. chains.

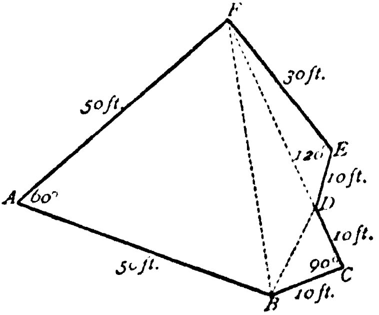
$= 1\cdot94375$ acres.
4

3·77500
40

31·000

∴ area = 1 a. 3 r. 31 p. Answer.

(2) The three alternate angles of a hexagon *ABCDEF*—namely *A*, *C*, and *E*—are 60°, 90°, and 120° respectively. The sides are $AB = AF = 50$ ft., $BC = CD = DE = 10$ ft., and $EF = 30$ ft. Find the area.



Draw the diagonals *FB*, *BD*, *DF*. Then the hexagon is divided into four triangles. In each of the triangles *ABF*, *BCD*, *DEF* two sides and the included angle are known; thus their areas can be found. And in the triangle *BDF* we can find the value of each of the sides, and so find the area of the triangle.

ABF is evidently an equilateral triangle, and *BCD* is right-angled.

$$\therefore \text{ area of triangle } ABF = \frac{50^2\sqrt{3}}{4} = 625\sqrt{3} \text{ sq. ft.} \quad . \quad . \quad . \quad (1).$$

$$\text{,,} \qquad BCD = \frac{100}{2} = 50 \text{ sq. ft.} \quad . \quad . \quad . \quad (2).$$

$$\text{,,} \qquad DEF = \tfrac{1}{2} \,.\, 10 \,.\, 30 \,.\, \frac{\sqrt{3}}{2} = \frac{300\sqrt{3}}{4}.$$

$$= 75\sqrt{3} \text{ sq. ft.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Sum of these three triangles $= 700\sqrt{3} + 50$ sq. ft.

We have now only to find the area of *BDF*.

Since *ABF* is equilateral, $BF = 50$ ft.

,, *BCD* is a right angle, $BD = 10\sqrt{2}$ ft.

and $DF = \sqrt{30^2 + 10^2 + 2\,.\,30\,.\,10\,.\,\tfrac{1}{2}} = \sqrt{900 + 100 + 300} = 10\sqrt{13}$ ft.

$\therefore$ area of *BDF*

$$= {}^{*}\tfrac{1}{4}\sqrt{(50 + 10\sqrt{2} + 10\sqrt{13})\ (50 + 10\sqrt{2} - 10\sqrt{13})\ (50 - 10\sqrt{2} + 10\sqrt{13})\ (-50 + 10\sqrt{2} + 10\sqrt{13})}.$$

$$= \frac{100}{4}\sqrt{(15 + 2\sqrt{26} - 25)\ (25 - 15 + 2\sqrt{26})}.$$

$$= 25\sqrt{(2\sqrt{26} - 10)\ (2\sqrt{26} + 10)}.$$

$$= 25\sqrt{104 - 100} = 50 \text{ sq. ft.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

$$\therefore \text{ area of hexagon} = 700\sqrt{3} + 100 \text{ sq. ft.}$$

If $\sqrt{3} = 1{\cdot}732$, $700\sqrt{3} = 173{\cdot}2 \times 7 = 1212{\cdot}4$.

$$\therefore \text{ area} = 1312{\cdot}4.$$

Thus area to the nearest sq. yd. $= 1312$. Answer.

---

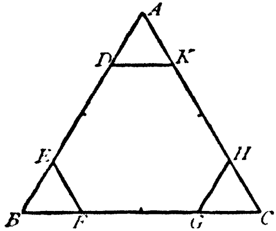
* In determining the area of a triangle when the sides are surds, it is more convenient to take the area of a triangle in the form:

$$\tfrac{1}{4}\sqrt{(a+b+c)\ (-a+b+c)\ (a-b+c)\ (a+b-c)},$$

which is identical with $\sqrt{s\ (s-a)\ (s-b)\ (s-c)}$, but does not employ the abbreviation s.

By a reference to the figure it will be seen that the angle *EDC* is of the kind called *re-entrant*. It is clear that polygons with re-entrant angles can be measured in exactly the same way as other polygons.

(3) Sometimes a polygon is formed by cutting off a corner or corners of a known triangle or quadrilateral. The following example will sufficiently illustrate this case.



The side of an equilateral triangle is 1 ft. Each of the sides is divided into four equal parts, and the nearest points of division joined. Find the area of the hexagon so formed.

We have to subtract from the original equilateral triangle the sum of three equilateral triangles, each of whose sides = ¼ ft. = 3 in.

$$\therefore \text{ area of triangle } ABC = \frac{12^2 \cdot \sqrt{3}}{4} = 36\sqrt{3} \text{ sq. in.}$$

$$,, \quad ADK = \frac{9 \cdot \sqrt{3}}{4}.$$

$$\therefore \text{ area of hexagon} = 36\sqrt{3} - \frac{27}{4}\sqrt{3}.$$

$$= \frac{117\sqrt{3}}{4} \text{ sq. in.}$$

This reduces to 50·66 sq. in. Answer.

(4) *Use of the Field-book.*—In practice a polygonal field is usually measured by the segments into which the longest diagonal is divided by the perpendiculars from the other angular points, and the lengths of these perpendiculars, which divide the field into trapeziums and right-angled triangles. The diagonal is called the *base-line*, and the perpendiculars are considered as *off-sets* from it. The method in which these are entered in the Field-book will appear from the following example.

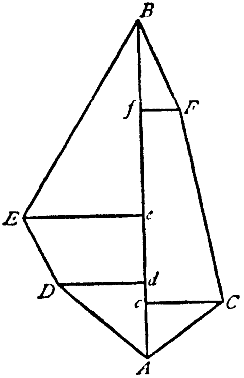
Find the area of a hexagonal field from the annexed measurements in the Field-book.

Here *AB* is the base-line, and the lengths of the off-sets to the other angular points, with their directions, are given at the sides, the middle figures giving the dis-

	Yards.	
	To *B*.	
	460	
	340	50 *F*
E 160	200	
D 120	110	
	70	100 *C*
	From *A*.	

tances from A at which the off-sets are measured. Hence the field may be represented by the following diagram.

$Ad = 110$ yds.
$Dd = 120$ „
$Ae = 200$ „
$Ec = 160$ „



$Ac = 70$ yds.
$Cc = 100$ „
$Af = 340$ „
$Ff = 50$ „
$AB = 460$ „

On both sides of the base-line there are two triangles and a trapezium to sum.

∴ area to right of base-line

$$= \tfrac{1}{2}(70 \times 100) + \tfrac{1}{2}(340 - 70)(100 + 50) + \tfrac{1}{2}(460 - 340)\,50 \text{ sq. yds.}$$
$$= 3500 + 20{,}250 + 3000 = 26{,}750 \text{ sq. yds.}$$

And area to left of base-line

$$= \tfrac{1}{2}(110 \times 120) + \tfrac{1}{2}(200 - 110)(120 + 160) + \tfrac{1}{2}(460 - 200)\,160.$$
$$= 6600 + 12{,}600 + 20{,}800 = 40{,}000 \text{ sq. yds.}$$

∴ total area of field

$= 66{,}750$ sq. yds. $= 13$ a. 3830 sq. yds. Answer.

EXAMPLES ON IRREGULAR POLYGONS

1. $ABCDE$ is an irregular pentagon. AD and CE intersect at right angles at O, and AB, BC are respectively parallel to EC, AD. Given $OA=76$ yds., $OC=35$ yds., $OD=30$ yds., $OE=40$ yds., find the area of the pentagon.

2. AD ($=160$ yds.) is the longest diagonal of a pentagonal field $ABCDE$. The feet of the perpendiculars on it from B, E, and C meet it at F, G, H, dividing it into four equal parts. Given $BF=56$ yds., $EG=36$ yds., $CH=34$ yds., find the area.

3. $ABCDE$ is a pentagon. The diagonal $BE=8$ chains, and the perpendiculars on it from A and C are 3 and 5 chains respectively; also the diagonal $CE=9$ chains, and the perpendicular on it from D is 2 chains. Find the area in acres, &c.

4. In a six-sided figure four of the sides in order are 20, 24, 15, and 18 ft. respectively. The perpendiculars on them from the intersection of the other two sides are in the same order 6, 26, 32, and 4 ft. Find the area.

5. Four sides, AB, BC, CD, DE, of a hexagon $ABCDEF$ subtend equal angles at the angular point F, and the distances FA, FB, FC, FD, FE are 30 ft., 36 ft., 27 ft., 20 ft., and 16 ft. respectively. Find the area, the angle at F being 120°.

6. In a hexagon $ABCDEF$, given the six sides $AB=93$ ft., $BC=65$ ft., $CD=45$ ft., $DE=39$ ft., $EF=16$ ft., $FA=12$ ft., and also given the diagonals $AC=34$ ft., $CE=42$ ft., $EA=20$ ft., find the area.

7. In a hexagonal figure, given the diagonals $AC=32$ yds., $CE=24$ yds., $EA=40$ yds., and the perpendicular from B on AC 17 yds., from D on CE 15 yds., and from F on EA 13 yds., find the area.

8. In the heptagonal field $ABCDEFG$, BC ($= 12$ poles) and AG ($= 17$ poles) are both perpendicular to AB. Given the perpendiculars from D, E, F on AB to be 18, 20, and 24 poles respectively, and the segments into which they divide AB to be 10, 9, 11, and 5 poles in order from A to B, find the area in acres, roods, and perches.

9. In a six-sided figure $ABCDEF$, AD cuts both FB and EC at right angles at G and H. Show that the area of the hexagon is the arithmetic mean between the rectangles contained by FB, AH, and EC, GD.

Find the area of a field of this shape, if $FB = 110$ yds., $EC = 99$ yds., $AG = 12$ yds., $GH = 43$ yds., $HD = 45$ yds.

10. In a hexagon, given the six sides and three alternate angles, find the area.

If $ABCDEF$ be the hexagon, let $AB = 5$ in., $BC = 15$ in., $CD = 10$ in., $DE = 4$ in., $EF = 8$ in., $FA = 12$ in., $A = 90°$, $C = 120°$, $E = 135°$. (The following results may be used to help the work: $\sqrt{19} = 4.359$, $\sqrt{5 + 2\sqrt{2}} = 2.798$, $\log 22.9935 = 1.36161$, $\log 1.1985 = .07864$, $\log 11.8015 = 1.07194$, $\log 57.011 = 1.75596$, $\log 9.9935 = .99972$.)

11. If, in the pentagon $ABCDE$, the angles at A and D are right angles, and the five sides in order are $AB = 92$ ft., $BC = 204$ ft., $CD = 180$ ft., $DE = 19$ ft., $EA = 69$ ft., find the area.

12. The longer diagonal AD of a hexagonal field $ABCDEF$ is 119 yds.; and it is divided by the perpendiculars let fall from F, B, E, C (which meet it in this order) into segments which have the ratio to each other of $2 : 3 : 7 : 1 : 4$. The perpendiculars from F, B, E, C are 15, 35, 32, and 30 yds. respectively. Find the area.

13. In a pentagon $ABCDE$, $A = 90°$, $C = 120°$, $B = 135°$; $AB = 5$ ft., $BC = 10$ ft., $CD = 10$ ft., $AE = 5$ ft. Find the area to nearest sq. ft.

14. In a hexagon $ABCDEF$, the angles at A, C, and E are right angles; and the sides have the following lengths: $AB = 1$ ft., $BC = 3$ ft., $CD = 4$ ft., $DE = 2$ ft., $EF = 3$ ft., $FA = 1$ ft. Show that the area is exactly 10 sq. ft.

15. In a pentagon $ABCDE$, the angles at A and C are respectively 90° and 120°; and the five sides are $AB = 4$ in., $BC = 5$ in., $CD = 1$ in., $DE = 3$ in., $EA = 2$ in. Find the area correct to one-thousandth part of a sq. in.

16. In a pentagon $ABCDE$, the angles at A and C are respectively 90° and 120°; and the five sides in order are $AB = 1$ ft., $BC = 1$ ft., $CD = 3$ ft., $DE = 5$ ft., $EA = 1$ ft. Find the area to nearest sq. in.

17. In the hexagonal field $ABCDEF$, $A = 90°$, $C = 60°$, $E = 120°$. Given $AB = EF = 6$ chains, $BC = DE = 2$ chains, $CD = 8$ chains, and $FA = 4$ chains, find the area to the nearest sq. pole.

18. In the pentagon $ABCDE$, if the diagonals AD, AC be drawn, AED, ADC, and ACB are right angles. If $AE = 12$ ft., $DE = 16$ ft., $DC = 15$ ft., $CB = 60$ ft., find the area. Also find AB.

19. In the pentagon $ABCDE$, the angles at A and D are right angles, and if EB be drawn, EBC is also a right angle. If $AB = 9$ in., $BC = 1$ ft. 8 in., $ED = 7$ in., and find the area.

20. A field $ABCDE$, whose area is 9750 sq. yds., is in shape a rectangle with the corner cut off by the side DE. If the field were a complete rectangle, the area would be 250 sq. yds. more. Given $AE = 150$ yds., and $CD = 40$ yds., find how much AE and CD must severally be produced to complete the rectangle.

21. A rectangular room has a corner cut off by the fireplace. If the two sides opposite to the fireplace are 22 ft. 10 in. and 15 ft. 3 in., and the two adjacent ones are 19 ft. 6 in. and 12 ft. 9 in., find the area of the floor.

22. A square whose area = 45 sq. in. has its sides trisected, and the nearest points of trisection joined, so as to cut off its corners. Find the area of the resulting octagon.

23. In an equilateral triangle ABC, whose side is a, the points D, E are taken in AB, so that $BE = AD = \frac{1}{5} AB$; F, G in BC, so that $BF = CG = \frac{1}{6} BC$; and H, K in AC,

so that $HC = KA = \frac{1}{3} AC$. Find the area of the hexagon $DEFGHK$.

24. An octagonal room is in shape a square with its four corners cut off evenly, so as to leave in each case half the side of the original square. The perimeter of the room is 60 ft. Find its area to a sq. ft.

25. $ABCD$ is a rectangle whose side $AB = 6$ in., and 11 in. From O, the middle point of AB, OE and OF are drawn, meeting AD and BC at angles of 30° and 45° respectively. Find the area of the pentagon $OFCDE$ to three decimal places of a sq. in.

26. Find in acres and sq. yds. the area of a field $ACEBFD$ from the following measurements in a field-book;

	Yards.	
	To B.	
	300	
F 60	250	
	200	80 E
D 100	120	
	60	75 C
	From A.	

27. Draw a plan of a field from the following measurements, and find its area in acres, roods, and poles.

	Yards.	
	To B.	
	340	
	210	120 E
C 80	30	80 D
	From A.	

28. Plan a field from the following notes, and find its area in acres, roods, and poles. (*Oxford Local.*)

	Yards	
	To *B*.	
	500	
E 120	420	30 *F*
C 160	120	160 *D*
	From *A*.	

29. Find the area of a field in acres, roods, and poles from the following notes:

	Chains.	
	To *B*.	
	16	
	12·45	7·6 *E*
D 6·35	8·5	
	3·75	4·8 *C*
	From *A*.	

30. Find the area from the following notes:

	Chains.	
	To *B*.	
	12·74	
G 3·68	10·22	
	8·96	4·38 *F*
		5·5 *E*
D 3·54		
C 2·75		
	From *A*.	

CHAPTER IV.—REGULAR POLYGONS

Section I.

[FORMULÆ:

In a regular polygon n = number of sides, a = the length of a side, r = radius of inscribed circle, R = radius of circumscribed circle.

$$r = \frac{a}{2\tan\frac{180°}{n}} \quad \ldots (1). \qquad R = \frac{a}{2\sin\frac{180°}{n}} \quad \ldots (2).$$

Also, if r be the radius of a circle,

$$\text{Side of inscribed polygon} = 2r\sin\frac{180°}{n} \quad \ldots\ldots (3).$$

$$\text{Side of circumscribed polygon} = 2r\tan\frac{180°}{n} \quad \ldots (4).]$$

26. Regular figures, as defined in the last chapter, are such as have all their sides equal and all their angles equal.

The regular figures of three and four sides (i.e. the equilateral triangle and the square) have been already discussed; it remains to discuss regular *polygons*, when the number of sides is greater than four. When, however, we talk of the "*regular polygon of n sides*," the equilateral triangle and the square are necessarily included, as particular cases in which $n = 3$ and $n = 4$.

27. Regular polygons all have this important property: that a circle can be inscribed in any regular polygon, and another circle can be described about it. These circles are known as the *inscribed* and *circumscribed circles*. They are

concentric, their common centre being the point at which the bisectors of the angles of the polygon meet. These properties are proved by Euclid only in the simpler cases of the square and regular pentagon; but his results may be easily extended to any regular polygon.

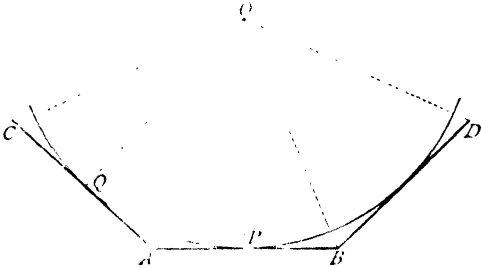


Fig. 1.

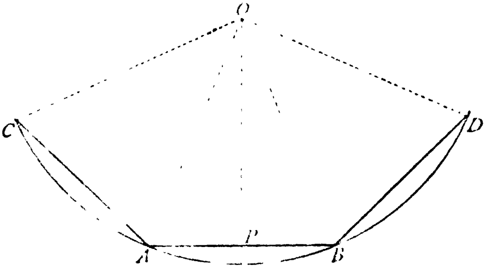


Fig. 2.

For let *CA*, *AB*, *BD* be three adjacent sides of any regular polygon. Let the bisectors of the angles at *A* and *B* meet in *O*, and join *CO*, *DO*. Then it may be proved, as in Euclid, iv. 13, that *CO*, *DO* bisect the angles at *C* and *D*, and that in consequence any straight line drawn from *O* to an angular point bisects the angle at it. Hence all the bisectors of the angles meet at *O*.

Next we can prove, as in Euclid, iv. 13 (fig. 1), that the perpendiculars from O on the sides OP, OQ are all equal, so that OP, OQ, &c., are radii of a circle which will touch all the sides; and, as in Euclid, iv. 14 (fig. 2), that the lines OA, OB, OC, &c., are all equal, so that they are radii of a circle which will pass through all the angular points.

28. *To find R, r, the radii of the circumscribed and inscribed circles.*

Since the sides AB, AC, AD . . . . are all equal, the arcs AB, AC, AD . . . . are also equal.

∴ also the angles subtended at the centre by these arcs, i.e. AOB, AOC . . . . are all equal.

$$\therefore \angle AOB = n^{\text{th}} \text{ part of two right angles} = \frac{360^\circ}{n}.$$

Now the triangles OAP and OBP are equal in all respects, having two angles equal in each, and a side common.

$$\therefore AOP = \frac{180^\circ}{n}, \text{ and } AP = \tfrac{1}{2} AB = \frac{a}{2}.$$

$$\therefore \text{(fig. 1)} \quad r = OP = \frac{AP}{\tan AOP} = \frac{a}{2 \tan \dfrac{180}{n}} \qquad (1).$$

$$\text{(fig. 2)} \quad R = OA = \frac{AP}{\sin AOP} = \frac{a}{2 \sin \dfrac{180}{n}} \quad . \quad . \qquad (2).$$

29. The angle of the regular polygon can be easily determined, for

$$\begin{aligned} CAB &= \text{twice } OAP. \\ &= \text{twice the complement of } AOP. \\ &= 2\left(90^\circ - \frac{180^\circ}{n}\right) = \frac{n-2}{n}\, 180^\circ \qquad (3). \end{aligned}$$

It should be noticed that the angle of the regular polygon depends only on the *number* of sides.

This angle may be used for determining some of the diagonals of the figure. In the case of a polygon with an even number of sides, the diameter of the circumscribed circle is the longest diagonal. But the determination of r and R are really the only problems of importance connected with the measurement of the lines of regular polygons.

Formulæ (1) and (2) may also be regarded from another standpoint. Suppose r the radius of a circle,

$$\therefore \text{ side of inscribed polygon} = 2r \sin \frac{180°}{n} \quad . \quad . \quad . \quad (3).$$

$$\text{side of circumscribed polygon} = 2r \tan \frac{180°}{n} \quad . \quad . \quad (4).$$

30. The three most important regular polygons are those with five, six, and eight sides respectively; i.e. the *pentagon*, the *hexagon*, and the *octagon*. Before proceeding to the case of examples, we will see what forms the formulæ for R and r take in these cases.

(1) In the *pentagon*, $R = \dfrac{a}{2 \tan 36°}$, $r = \dfrac{a}{2 \sin 36°}$.

By calculating the sine and tangent of 36°, we obtain that $r = a \times .688191$, $R = a \times .850651$.

(2) In the *hexagon*, $r = \dfrac{a}{2 \tan 30°}$, $R = \dfrac{a}{2 \sin 30°}$.

Here $r = \dfrac{\sqrt{3} \cdot a}{2}$, $R = a$; i.e. the radius of the circumscribed circle = side of the hexagon, as Euclid proves (iv. 15).

(3) In the *octagon*, $r = \dfrac{a}{2 \tan 22° \, 30'}$, $R = \dfrac{a}{2 \sin 22° \, 30'}$.

Here $r = \dfrac{a(\sqrt{2} + 1)}{2}$; while by calculating sin 22° 30′, we shall obtain $R = a \times 1.306563$.

Examples.—(1) Find the radius of the circle inscribed in a regular hexagon whose side $=6$ in.

From the last section $r=\frac{\sqrt{3}\,.\,a}{2}=\frac{\sqrt{3}\,.\,6}{2}=3\sqrt{3}$ in.

$=3\times1{\cdot}732=5{\cdot}196$ inches. Answer.

(2) Find side of a regular pentagon inscribed in a circle whose diameter $=4$ ft. 2 in.

$$a=2r\sin 36^\circ=50\sin 36^\circ.$$

If we calculate sin 36°, we shall find it $={\cdot}587785$.

$$\therefore\ a=50\times{\cdot}587785=29{\cdot}38925 \text{ inches.}$$

(It is not usually needful to calculate to so many decimal places. If the above answer were to be *correct to an inch*, sin 36° might be taken as ·59. It must be noticed, as a general rule in all approximations, that the last figure has to have 1 added on if the figure following be greater than 5.)

(3) The perimeter of a regular octagon inscribed in a circle is 20 in. Find the perimeter of an equilateral triangle circumscribed to the same circle correct to two decimal places of an inch.

Side of octagon $=\frac{20}{8}=\frac{5}{2}$ in.

Radius of circumscribed circle $=\frac{5}{2}\times1{\cdot}306563$. (See Art. 30.)

Side of circumscribed equilateral triangle

$$=2\,.\,\tfrac{5}{2}\times1{\cdot}306563\tan 60^\circ=5\sqrt{3}\times1{\cdot}306563.$$

$$\therefore\ \text{perimeter of triangle}=15\sqrt{3}\times1{\cdot}306563.$$

If we take $\sqrt{3}=1{\cdot}732$, and take the other decimal to four places, we shall obtain perimeter $=33{\cdot}945468$ in.

$\therefore$ correct to two decimal places, answer $=33{\cdot}95$ in.

EXAMPLES ON THE LENGTHS OF REGULAR POLYGONS

1. If each angle of a regular polygon is 162°, find the number of sides.

2. If each angle of a regular polygon is 165°, find the number of sides.

3. The angles in one regular polygon are twice as many as the angles in another regular polygon, and an angle of the former is to an angle of the latter as 3 : 2. Find the number of sides. (*Sandhurst.*)

4. The angle of a regular polygon is $\frac{3}{5}$ of that of another regular polygon with three times the number of sides. Find the number of sides in first polygon.

5. The side of an equilateral triangle is 20 ft. Find the numerical value of the radius of the circle circumscribing the triangle. (*Sandhurst.*)

6. Find the side of a regular pentagon which is described about a circle of 2 ft. radius. Answer to $\frac{1}{10}$ of a foot.

7. Find the perimeter of a regular hexagon which is inscribed in a circle whose diameter is 3 ft.

8. The side of a regular dodecagon is 8 ft. Find the diameters of the inscribed and circumscribed circles.

9. Find to five decimal places the side of a regular heptagon inscribed in a circle of radius 50 in., given that $\sin 25^\circ\ 42' = \cdot 4336591$, $\sin 25^\circ\ 43' = \cdot 4339212$.

10. The radii of the inscribed and circumscribed circles of a regular octagon are to each other as $\sqrt{2+\sqrt{2}} : 2$.

11. The diameter of a circle is 12 ft. Find the side of the inscribed decagon to three places of decimals.

12. Show that the length of the side of an equilateral triangle inscribed in a circle is to that of a side of the square inscribed in the same circle as $\sqrt{3} : \sqrt{2}$. (*Sandhurst.*)

13. An equilateral triangle and a square have the same perimeter. Find the ratio of the diameters of the circles inscribed in them.

14. Compare the perimeters of the inscribed regular hexagon, and the circumscribed equilateral triangle, of a given circle.

15. Compare the radius of the inscribed circle of a regular hexagon with that of the circumscribed circle of a regular pentagon, the side of the hexagon being to that of the pentagon as $\sqrt{3} : \sqrt{2}$.

16. Find the longest diagonal of a regular octagon whose side is equal to the diagonal of a square 100 ft. in area. Calculate the answer to two decimal places of a foot.

17. In a regular pentagon, whose side is a, find the length of the straight lines joining any two angular points not adjacent. Example: Let $a = 2$ ft.

18. In a regular pentagon, the radius of whose inscribed circle is 15 in., find correct to three decimal places the number of inches in the perpendicular let fall from any angular point on the opposite side.

19. In a regular pentagon the angular points are joined by straight lines, which form by their intersection another pentagon. Compare the perimeter of this pentagon with the former.

20. Find in feet and inches, correct to $\frac{1}{100}$ inch, the length of the straight line joining the middle points of two opposite sides of a regular hexagon whose side is 1 ft.

21. The perimeter of a regular hexagon is to that of a regular octagon as $2 : 3$. Find the ratio of the radius of the circle inscribed in the hexagon to that of the circle described about the octagon (to two decimal places).

22. Sixty-one coins of equal area are placed, touching one another, so that the outside presents the appearance of a regular hexagon. Find the number of coins in the outside edge.

23. A circle of known radius has a square circumscribing it, and a regular pentagon inscribed in it. Compare the radius of the circle described about the square with that of the circle inscribed in the pentagon.

24. Prove that the square on the side of an inscribed pentagon is equal to the sum of the squares of the sides of a hexagon and decagon inscribed in the same circle. (*Sandhurst.*)

25. The angle of a regular octagon is the arithmetical mean between the angles of the regular hexagon and the regular dodecagon.

26. In a circle of known radius there is inscribed a regular hexagon; in this hexagon a second circle is inscribed; and in the second circle a second hexagon. Find the value of the radius of a circle inscribed in the second hexagon.

27. An equilateral triangle has a circle described about it, and another circle inscribed in it. A hexagon is described about the first circle, and another hexagon is inscribed in the second circle. Show that the side of the first hexagon is to the side of the second as $4 : \sqrt{3}$.

Section II.—Areas

[FORMULÆ:

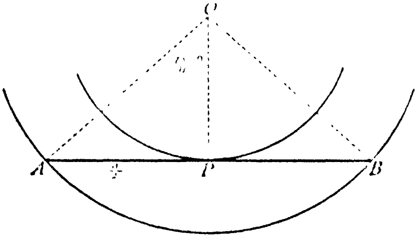
Area of regular polygon of n sides,

(1) In terms of the side $a = \frac{na^2}{4} \cot \frac{180°}{n}$.

(2) In terms of the radius $r = nr^2 \tan \frac{180°}{n}$.

(3) In terms of the radius $R = \frac{nR^2}{2} \sin \frac{360°}{n}$.]

31. Let AB be the side of a regular polygon of n sides.



Take O, the centre of the inscribed and circumscribed circles. Then OP, the radius of the inscribed circle, is perpendicular to AB. (Euc. iii. 18.) The radii OA, OB, &c., being all equal, and containing equal angles (each $= \frac{360°}{n}$), evidently divide the polygon into n equal triangles.

$\therefore$ area of polygon $= n$ times triangle AOB.

2 2

But $r = \dfrac{a}{2 \tan \frac{180°}{n}} = \frac{a}{2} \cot \frac{180°}{n}$.

$\therefore$ area of polygon $= \frac{na^2}{4} \cot \frac{180°}{n}$ . . . (1).

Again, $a^2 = 4r^2 \tan^2 \frac{180°}{n}$.

Substituting, we obtain the area in terms of r, the radius of the inscribed circle.

$$\text{Area} = \frac{n}{4} \cdot 4r^2 \cdot \tan^2 \frac{180°}{n} \cot \frac{180°}{n}.$$

$$= nr^2 \tan \frac{180°}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Again, $R = \frac{a}{2 \sin \frac{180°}{n}}$. $\therefore \ a^2 = 4R^2 \sin^2 \frac{180°}{n}$.

Substituting again in (1), we obtain the area in terms of R, the radius of the circumscribed circle.

$$\text{Area} = \frac{n}{4} \cdot 4R^2 \sin^2 \frac{180°}{n} \cot \frac{180°}{n}.$$

$$= \frac{n}{4} \cdot 4R^2 \sin \frac{180°}{n} \cos \frac{180°}{n}.$$

$$= \frac{nR^2}{2} \sin \frac{360°}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

32. All three of these results could be readily deduced from the figure without employing the formulæ for the two radii. For instance, let us deduce (3) directly.

$$\text{Area of triangle } AOB = \tfrac{1}{2} AO \cdot OB \sin AOB.$$

$$= \frac{R^2}{2} \sin \frac{360°}{n}.$$

$$\therefore \text{ area of polygon} = \frac{nR^2}{2} \sin \frac{360°}{n}.$$

If r be the radius of any circle, it follows from formulæ (2) and (3) that:

Area of inscribed polygon of n sides $= \frac{nr^2}{2} \sin \frac{360°}{n}$.

Area of circumscribed polygon of n sides $= nr^2 \tan \frac{180°}{n}$.

33. We will now see what these formulæ become in the case of the three more important figures.

(1) In the *pentagon*.

Area in terms of side $a = \frac{5a^2}{4} \cot 36^\circ$.

This is found by calculation to be:

$$\text{Area} = a^2 \times 1.720477.$$

Thus the area of a regular pentagon is found by multiplying the square of its side by 1.720477.

Area in terms of radius of inscribed circle

$$= 5r^2 \tan 36^\circ = r^2 \times 3.632713.$$

Area in terms of radius of circumscribed circle

$$= \tfrac{5}{2} R^2 \sin 72^\circ = R^2 \times 2.377641.$$

(2) In the *hexagon*.

Area in terms of the side a (or radius of circumscribed circle, which is equal to it)

$$= \frac{6a^2}{4} \cot 30^\circ = \frac{3\sqrt{3}}{2} a^2.$$

Area in terms of radius of inscribed circle

$$= 6r^2 \tan 30^\circ = 2\sqrt{3} \,.\, r^2$$

(3) In the *octagon*.

Area in terms of side $a = \frac{8a^2}{4} \cot 22^\circ 30'$.

$$= 2a^2 (\sqrt{2} + 1).$$

Area in terms of $r = 8r^2 \tan 22^\circ 30' = 8 (\sqrt{2} - 1) r^2$.

„ $R = 4R^2 \sin 45^\circ = 2\sqrt{2} \,.\, R^2$.

Examples.—(1) Find to the nearest square inch the area of a regular pentagon, the diameter of whose inscribed circle is 3 ft. 4 in.

Area of pentagon $= r^2 \times 3{\cdot}632713$ (see above), and here $r = 20$in.

∴ area $= 400 \times 3{\cdot}632713$ sq. in.

$= 1453{\cdot}0852$ sq. in. $=$ 10 sq. ft. 13 in. Answer.

(The answer would have the required correctness if the area of the pentagon were taken as $r^2 \times 3{\cdot}633$. For the difference between this and the real value is less than $r^2 \times \cdot 0003 = 400 \times \cdot 0003$ sq. in. $= \cdot 12$ sq. in.)

(2) Find to the nearest half-inch the radius of a circle such that the area of the regular hexagon inscribed in it is 80 sq. ft. less than that of the equilateral triangle described about it.

Let r = the radius.

$$\text{Area of equilateral triangle} = 3\sqrt{3}\,.\,r^2.$$

$$\text{,, \quad regular hexagon} = \frac{3\sqrt{3}}{2}r^2.$$

$$\therefore\ 3\sqrt{3}r^2 - \frac{3\sqrt{3}r^2}{2} = 80 \text{ sq. ft.}$$

$$3\sqrt{3}r^2 = 160.$$

$$9r^2 = 160\sqrt{3}$$

$$\frac{9r^2}{16} = 10\sqrt{3} = 17\cdot32.$$

$$\therefore\ \frac{3r}{4} = 4\cdot16 \text{ ft.}$$

$$r = 5\cdot54 \text{ ft.} = 5 \text{ ft. } 6\tfrac{1}{2} \text{ in.}\quad \text{Answer.}$$

(3) The perimeter of a regular octagonal room is 76 ft. Find the cost of linoleum for the floor at 2*s.* 9*d.* per sq. yd.

$$\text{Side of octagon} = \tfrac{76}{8} = \tfrac{19}{2} \text{ ft.}$$

$$\therefore\ \text{area of floor} = 2\,.\,\frac{361}{4}\,.\,(\sqrt{2}+1) \text{ sq. ft.}$$

$$= \tfrac{361}{2}(2\cdot414) = 361 \times 1\cdot207.$$

$$= 435\cdot727 \text{ sq. ft.}$$

$$\text{Cost} = \overset{d}{\tfrac{33}{9}} \times 435\cdot727 = \tfrac{11}{3} \times 435\cdot727d.$$

$$= 1597\cdot666d. = £6\ 13s.\ 1\tfrac{2}{3}d.\quad \text{Answer.}$$

(4) The longest diagonal of a courtyard, which is in shape a regular dodecagon, is 18 yds. Find the side of a square courtyard of equal area.

The longest diagonal is the diameter of the circumscribed circle.

$$\therefore\ \text{area of courtyard} = \frac{12R^2}{2}\sin 30^\circ, \text{ where } R = 9 \text{ yds.}$$

$$\therefore\ \text{area} = \frac{12\,.\,81}{2}\,.\,\tfrac{1}{2} = 243 \text{ sq. yds.}$$

Side of square whose area is 243 sq. yds.

$$= \sqrt{243} = 9\sqrt{3} = 9\,(1\cdot732).$$

$$= 15 \text{ yds. } 1 \text{ ft. } 9 \text{ in.}\quad \text{Answer.}$$

EXAMPLES ON THE AREA OF REGULAR POLYGONS

1. Find the number of sq. inches in a regular pentagon whose side is 10 in.

2. Find correct to $\frac{1}{10}$ of an inch the area of a regular hexagon, inscribed in a circle whose radius is 8 yds.

3. Find the area of a regular octagon, the radius of whose inscribed circle is 25 in.

4. Find the area of a regular pentagon, the diameter of whose inscribed circle is 4 ft. 2 in., to nearest sq. in.

5. Find to a sq. in. the area of a regular decagon, the diameter of whose inscribed circle is 3 ft. 10 in.

6. Find the area of a regular dodecagon, the diameter of whose inscribed circle is 1 ft. Answer to three decimal places of a sq. in.

7. Find the difference in area between a heptagon whose side is 10 in., and the regular pentagon, the radius of whose inscribed circle is 10 in.

8. The ratio of the inscribed to the circumscribed dodecagon of any circle is $2 + \sqrt{3} : 4$.

9. If the area of the inscribed quindecagon of a circle be 100 sq. ft., find the area of the circumscribed dodecagon, correct to a sq. ft.

10. The perimeters of a decagon and dodecagon are as 15 : 12. Compare their areas.

11. Show that the area of a dodecagon inscribed in a circle is three-quarters the area of the square described about the same circle.

12. Find in square links the area included between the inscribed regular decagon and the circumscribed regular pentagon of a circle whose diameter is 80 links. Given $\sin 36^\circ = .58778$, and $\tan 36^\circ = .72654$.

13. The area of the dodecagon inscribed in a given circle is 240 sq. in. Find to the nearest sq. in. the area of the inscribed pentagon.

14. Given sin 30′ = .0087265, find to four places of decimals the ratio of the area of a regular polygon of 720 sides to the square of the radius of the circle described about it.

15. The area of the circumscribed regular polygon of *n* sides is a harmonic mean between the areas of the inscribed regular polygon of *n* sides, and of the regular polygon of half the number of sides described about the same circle. (*Sandhurst.*)

16. The alternate angular points of a regular hexagon (side *a*) are joined so as to form another regular hexagon. Compare its area with that of the former.

17. A field is in shape a regular hexagon. If the rent, at £2 10*s.* per acre, is £16 4*s.* 9*d.*, find in chains the length of the hedge surrounding it.

18. Find the cost of paving with stone, at 3*s.* 9*d.* per sq. yd., the floor of an octagonal church-tower of regular form, if the internal perimeter of the tower be 80 ft.

19. If the cost of running a fence round a uniform hexagonal field be £11, at 8*d.* per yd., find the rent, at £2 13*s.* 4*d.* per acre, to the nearest farthing.

20. Cost of paving with tiles, at 4*s.* 7*d.* per sq. yd., a regular hexagonal courtyard whose greatest breadth is 24 ft.

21. A uniform octagonal table is 32 in. at its widest part. Find the area of the surface correct to a square inch, and find the perimeter of a square table of equal area.

22. A square flower-bed, whose area is 128 sq. ft., is changed in shape into a regular octagon, four of whose angular points are the same as before, and the other four equidistant from them. Find to the nearest sq. ft. by how much the bed is increased in area.

23. The base of a stone pillar is a regular hexagon, the side of which is 2 ft. 6 in. Compare the area on which it stands with (i) that of a square base, (ii) that of an octagonal base, both having the same perimeter.

24. Cost of hurdles, 6 ft. long each, at 9*d.* a hurdle, to surround a field which is in shape a regular dodecagon, the area being 5 a. 3 r. 5 p. 9 sq. yds.

25. A rose-window consists of 7 panes, all equal and regular hexagons, one being in the centre, and the other six ranged round it with their sides continuous. If the extreme height of the window is $7\frac{1}{2}$ ft., find its area.

26. Find the area of a regular heptagon described about a circle whose radius is 70 ft. Given $\log 7 = .8450980$, $L \tan 25^\circ 42' 51\frac{3}{7}'' = 9.6826636$, $\log 16518 = 4.2179575$.

27. A farmer has 180 hurdles, each 2 yds. long, which he wishes to arrange so as to surround as large an area as possible. In what shape had he best arrange them? and what will be the number of sq. yds. in the enclosed area? Given $L \cot 1^\circ = 11.7580785$, $\log 2 = .3010300$, $\log 3 = .4771213$, $\log 10312 = 4.0133429$, $D = 421$.

28. A circular cricket-ground is roped round by posts at regular intervals. The amount of rope used is 480 yds., and the number of posts 120. Find approximately, in acres and yards, the area of the ground enclosed. Given $L \cot 1^\circ 30' = 11.58193$, $\log 2 = .30103$, $\log 3 = .47712$, $\log 18330 = 4.26316$.

29. A circular space, 12,100 sq. yds. in extent, is to be surrounded by posts and chains. If the number of the posts be 100, find what length of chain will be required. Given $\log 2 = .30103$, $\log 11 = 1.04139$, $L \tan 1^\circ 48' = 8.49729$, $\log 39 = 1.59106$.

CHAPTER V.—THE CIRCLE

Sec. I.—Circumference and Chords

[FORMULÆ (radius $= r$):

(1) Circumference of circle $= 2\pi r$

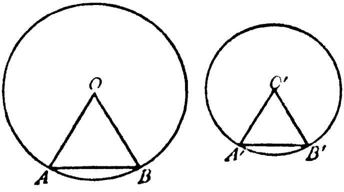
(2) Arc of circle $= r\theta$.]

34. The consideration of the regular polygon naturally leads on to that of the circle. In the last chapter we were led to the properties of the regular polygon by the help of the inscribed and circumscribed circles. In the present chapter we shall arrive at the properties of the circle by the help of the inscribed and circumscribed polygons.

We must assume that, if we inscribe in or describe about a circle a regular polygon of n sides, then by increasing the number of sides we can make the perimeter of either polygon differ by as small a quantity as we please from the circumference of the circle; so that, when the number of sides is increased indefinitely (that is, without limit), the perimeter of either polygon may be considered identical with the circumference of the circle.

35. *The circumferences of circles are proportional to their radii.*

Take two circles, whose radii are r_1 and r_2. Let AB and $A'B'$ (a_1 and a_2) be the sides of regular polygons of the same number of sides inscribed in each. Take O, O', the centres, and join OA, OB, $O'A'$, $O'B'$.



Then, since the angle

$$AOB = \text{angle } A'O'B' \text{ (for each} = \frac{360^\circ}{n}\text{), and}$$

$$AO : OB :: A'O' : O'B' \text{ (for } AO = OB \text{ and } A'O' = O'B'\text{).}$$

∴ the triangles AOB, $A'O'B'$, are similar (Euclid, vi. 6),

and $$OA : O'A' :: AB : A'B'.$$

i.e. $$r_1 : r_2 :: a_1 : a_2.$$
$$:: na_1 : na_2.$$

Thus the radii of the circles are proportional to the perimeters of the inscribed polygons.

Now let n be increased without limit, and the perimeters of the polygons will become ultimately identical with the circumferences of the circles.

∴ the radii of the circles are proportional to their circumferences. Q.E.D.

Thus the ratio of the circumference of a circle to its radius, and consequently to its diameter, is constant. The ratio of the circumference to the diameter is denoted by π. Thus, if r be the radius,

$$\text{Circumference of circle} = 2\pi r \quad . \quad . \quad . \quad (1).$$

The value of π cannot be exactly determined, but it may be calculated to any degree of approximation required. Its value to seven places of decimals is 3.1415927; but for purposes of ordinary problems $\frac{22}{7}$ is a sufficient approximation. If greater accuracy be required, π may be taken as 3.1416.

36. *Length of an arc of a circle.*

Let the arc subtend an angle θ at the centre. Then (by Euclid, vi. 33) the arc has the same ratio to the circumference that the angle θ has to four right angles.

If then θ be the circular measure of the angle,

$$\text{arc} : \text{circumference} :: \theta : 2\pi.$$

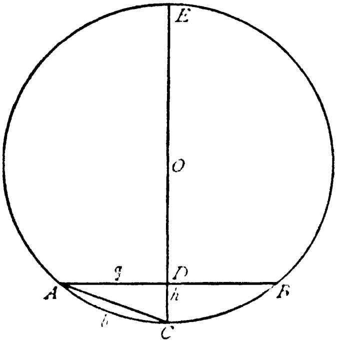
$$\therefore \text{ arc} = \frac{\theta \times 2\pi r}{2\pi} = r\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

If the number of degrees in the angle be given ($A°$),

$$\therefore \text{ arc} = \frac{A^\circ}{180^\circ} \times \pi r.$$

37. *Chord of a circle.*

Let AB be any chord ($= a$), and $EODC$ the diameter drawn at right angles to it.



Then CD is called the *height* (h) *of the arc AB.*

AC is called the *chord of half the arc* (b).

Since EC, AB meet in a circle at D (Euc. iii. 35),

$$AD \,.\, DB = CD \,.\, DE.$$

$$\therefore \; AD^2 = CD \,.\, DE.$$

$$\frac{a^2}{4} = h\,(2r - h).$$

$$\therefore \; a^2 = 4h\,(2r - h) \quad . \quad . \quad . \quad . \quad (3).$$

Also $b^2 = \dfrac{a^2}{4} + h^2 = h\,(2r - h) + h^2.$

$$\therefore \; b^2 = 2hr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Thus both the chord of an arc, and the chord of half an arc, can be expressed in terms of the height.

These formulæ need not be remembered, as they are readily deduced from the above property of the circle.

Examples.—(1) The diameter of a circle is 2 ft. 4 in. Find the circumference.

$$\therefore \text{ circumference} = 2\pi r = \pi \times 28 \text{ inches.}$$

If we take $\pi = \frac{22}{7}$, we obtain circumference $= \frac{22}{7} \times 28$.

$$= 88 \text{ in.} = 7 \text{ ft. } 4 \text{ in.}$$

(This is exact enough for ordinary calculations. But the student must observe carefully that *the exactness of the answer is delusive.* If we take $\pi = 3{\cdot}1416$, we shall obtain circumference $= 87{\cdot}9648$ in. Thus the former answer is too great by about ·0352 in., though correct to $\frac{1}{10}$ of an inch.)

(2) A coach-wheel is 5 ft. 3 in. in diameter. Find how often it turns round in a journey of 5 miles.

$$\text{Circumference of wheel} = \pi \times 63 \text{ in.}$$

$$= \tfrac{22}{7} \times 63 = 198 \text{ in.}$$

Number of inches in 5 miles $= 5 \times 1760 \times 3 \times 12$.

$$\therefore \text{ number of times wheel turns round} = \frac{5 \times \overset{160}{\cancel{1760}} \times 3 \times \overset{2}{\cancel{12}}}{\underset{18}{\cancel{198}}}.$$

$$= 1600 \text{ times.}$$

(3) At what time between 2 and 3 o'clock are the hands of a watch at an angle of 100°?

The two hands at an angle of 100° will enclose between them an arc $= \frac{100}{360} = \frac{5}{18}$ of the whole circumference.

$\therefore$ the hands are $\frac{5}{18}$ of $60 = 16\frac{2}{3}$ minutes apart.

Let x = number of minute-divisions passed through by the minute-hand before reaching the stated position.

$\therefore \frac{x}{12}$ = number of minute-divisions passed by hour-hand, and the minute-hand passes $10 + 16\frac{2}{3} = 26\frac{2}{3}$ more than the hour-hand.

$$\therefore x - \frac{x}{12} = 26\frac{2}{3}.$$

$$12x - x = 320.$$

$$\therefore x = \frac{320}{11} = 29\frac{1}{11} \text{ minutes past 2.} \quad \text{Answer.}$$

(4) The chord of an arc of a circle = 2 ft. 8 in.; the chord of half the arc = 1 ft. 8 in. Find the radius.

Here $AB = 2$ ft. 8 in.

$\therefore DB = 1$ ft. 4 in. $= 16$ in.

$BC = 20$ in.

$\therefore DC = \sqrt{20^2 - 16^2} = 12$ in.

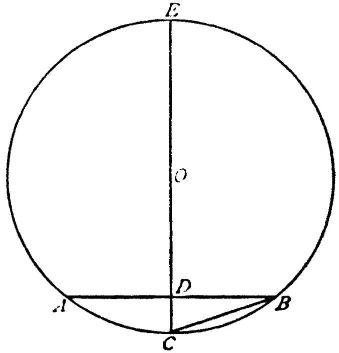
$ED \cdot DC = DB^2.$

$\therefore ED = \frac{16^2}{12} = \frac{64}{3}$ in.

$\therefore 2r = EC = \frac{64}{3} + 12.$

$= \frac{100}{3}$ in.

$\therefore r = \frac{50}{3} = 16\frac{2}{3}$ in. Answer.



N.B.—It is important to understand how far the approximation $\pi = \frac{22}{7}$ will give correct results. No *exact* general rule can be given, but usually *the first three figures* of an answer so obtained will be correct, and the error will commence in the fourth figure.

EXAMPLES ON THE CIRCUMFERENCE, Etc., OF A CIRCLE

[N.B.—$\pi = \frac{22}{7}$, except where otherwise stated.]

(a) Circumference

1. Find the circumference of a circle whose diameter is 13 ft. 5 in., and the diameter of a circle whose circumference is 15 ft. $8\frac{3}{8}$ in.

2. Find the circumference of a circle whose diameter is 3 fur. 37 po., and the radius if the circumference is 100 ft. ($\pi = 3.1416$.)

3. How much error, as far as five places of decimals, is involved in taking $\pi = \frac{22}{7}$ exactly? What will this error amount to if the diameter of the circle is 100 miles?

4. Show that if the radius of one circle is to the diameter of another as 5 : 12, and the circumference of the second is to the diameter of a third as 132 : 49, then the radii of the three circles are in arithmetical progression, assuming $\pi = \frac{22}{7}$.

5. The diameter of the earth may be considered 7925.6 miles. Show that the distance round the equator is barely 1 mile short of 24,900 miles, and that 1 degree at the equator contains rather more than 69 miles. ($\pi = 3.14159$.)

6. How often does a coach-wheel 3 ft. 9 in. in diameter turn round in a journey of 50 miles?

7. The large wheel of a bicycle is 50 in. in diameter, and the small wheel 9 in. How many times will the small wheel turn round more than the big wheel in going a journey of 10 miles?

8. How great a distance has been travelled by a coach, a wheel of which has turned round 7812 times, the length of each spoke of the wheel being 20 in.?

9. What is the height of a wheel which turns round 12,960 times in 3 hours, when the carriage is running at the rate of 1 mile per 8 minutes?

10. The total breadth of a gold ring is .8 in., and its inner circumference = 2.387616 in. Find the thickness of the gold. ($\pi = 3.1416$.)

11. Show that the perimeter of a regular hexagon is to the circumference of its circumscribed circle as 3 : π.

12. Show that the circumference of a circle lies between the perimeters of the inscribed and circumscribed regular polygons of 24 sides.

13. The perimeters of a regular octagon and a regular decagon are to each other as 4 : 5. If the circumference of the circle inscribed in the octagon be 10.98 in., find that of the circle described about the decagon.

14. A circular pathway surrounds a circular grass-plot. The inner circumference is 104 yds. 2 ft. 2 in., and the outer 125 yds. 2 ft. Find the breadth of the pathway. (π 3.1416.)

15. Find the cost of a fringe bordering a circular table whose diameter is 19 in., at 5*s.* 3*d.* a yard.

16. Find the cost of building a stone rim round a circular pond whose greatest breadth is 50 ft., at 4*s.* 9*d.* per yd.

17 If the cost of running palings round a circular piece of ground be £20 6*s.* 8*d.*, at 1*s.* 4*d.* per yd., find its greatest breadth.

18. The diameter of a running-ground is 476 ft. How many times will a bicycle-wheel 4 ft. 8 in. in diameter turn round in going 5 times round it?

(b) Arc

19. Find the length of that part of a circular railway-curve which subtends an angle of $22\frac{1}{2}^\circ$ to a radius of a mile. (*Sandhurst.*)

20. Find the distance in miles between any two places on the equator which differ in longitude by 6° 18′, assuming the earth's equatorial diameter to be 7925.6 miles. ($\pi = 3.1416$.) (*Sandhurst.*)

21. A pendulum 4 in. long swings through an arc of 10°. Find the length of the portion of a circumference it traces out, and the distance between the two furthest points reached by the pendulum. ($\pi = 3.1416$, $\sin 5° = .0872$.)

22. The hands of a clock form an angle 92° 30′, the minute-hand being nearest to the figure 12. If the minute-hand has an arc of 30° to pass through before reaching the hour, what is the time?

23. At what times between 6 and 7 o'clock are the hands of a clock at an angle of 70°?

24. Two places on the equator are 150 miles apart. What is their difference in longitude? (Diameter of earth is 7925.6 miles, $\pi = 3.14159$.)

(c) Chord

25. The chord of an arc is 8 ft., and the height of the arc 2 ft. What is the radius of the circle? (*Sandhurst.*)

26. Given the chord = 3 in., and the radius = $2\frac{1}{2}$ in., find height of arc.

27. Given chord of arc 4.8 in., and chord of half the arc 2.6 in., find the radius.

28. Given height of arc $\frac{17}{16}$ in., and chord of half the arc $3\frac{3}{8}$ in., find the radius.

29. Given chord of half the arc = 2 ft. 11 in., and circumference of circle = 15 ft. $3\frac{1}{3}$ in., find the height.

30. Given height of arc 4 in., and circumference of circle $40\frac{6}{7}$ in., find the chord.

31. Show that the formulæ in Art. 37 are directly deducible from Euclid vi. 8.

Section II.—Area

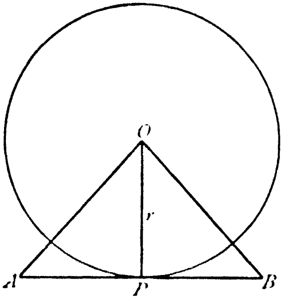
[FORMULÆ :

(1) Area of circle $= \pi r^2$.

(2) " sector $= \frac{r^2\theta}{2}$.

(3) " segment $= \frac{r^2}{2}(\theta - \sin\theta)$.]

38. *To find the area of a circle whose radius is r.*



Let AB be the side of a circumscribed polygon of n sides.

$\therefore$ area of triangle $AOB = \frac{1}{2} AB \times OP = \frac{1}{2} AB \times r$.

Area of polygon

$$= \frac{n \cdot AB}{2} \times r.$$

$$= \tfrac{1}{2} \text{ perimeter} \times r.$$

Now if the number of sides be indefinitely increased, the perimeter ultimately becomes the circumference of the circle, and the area of the polygon ultimately becomes the area of the circle.

$$\therefore \text{ area of circle} = \tfrac{1}{2} \text{ circumference} \times r.$$

$$= \tfrac{1}{2} \cdot 2\pi r \times r.$$

39. *To find the area of the sector of a circle.*

Let θ be the circular measure of the angle of the sector.

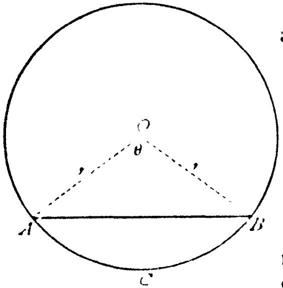
$$\therefore \text{ sector : area of circle : : } \theta : 2\pi.$$

$$\text{sector} = \frac{\theta \cdot \pi r^2}{2\pi} = \frac{r^2\theta}{2}$$

Since arc $= r\theta$, we may say that:

Area of sector $= \frac{1}{2}$ arc × radius.

40. *To find the area of the segment of a circle.*



Let r and θ mean the same as before.

$\therefore$ segment ACB

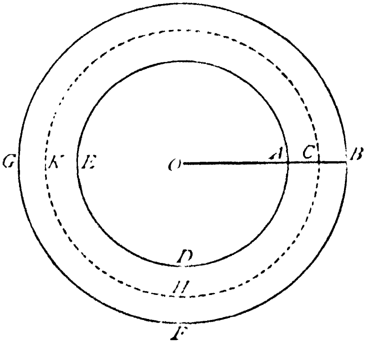
sector $OACB$ − triangle AOB.

$$r^2\theta \quad r^2 \sin \theta$$

Thus the area of a segment is most naturally expressed in terms of the angle its chord subtends at the centre. If the chord of the segment be given and the radius, or the chord and height of segment, the formula becomes too cumbrous to be of much use.

41. *The circular ring.*

The space included between two concentric circles is called a circular ring. If r_1, r_2 are the radii of the two circles, area of ring $= \pi\ (r_2^2 - r_1^2)$. The ring may be regarded as a rectangle bent round. On this view AB is called the *breadth* of the ring, and the circumference CHK, half-way between the outer and inner circumferences, is called its *length*.



Also area of ring   length × breadth.

For length $= 2\pi \,.\, OC = 2\pi \,.\, \dfrac{r_1 + r_2}{2} = \pi\,(r_1 + r_2)$,

and breadth $= OB - OA \quad r_2 - r$

$\therefore$ length × breadth

$$= \pi\,(r_2 + r_1)\,(r_2 - r_1) = \pi\,(r_2^2 - r_1^2) = \text{area of ring.}$$

Examples.—(1) A farmer wishes to lay out £10 in enclosing with a fence as large a piece of ground as possible. If fencing cost 7*d.* per yd., what is the largest area he can enclose?

A circle is the figure which encloses most space with a given perimeter; so that the enclosure will be a circle.

Circumference of circle $= \frac{2400}{7}$ yds.

$2\pi r = \frac{2400}{7}$. $\quad \therefore\ r = \frac{2400}{7} \times \frac{7}{44} = \frac{600}{11}$ yds.

$\therefore$ area $= \pi r^2 = \frac{22}{7}\left(\frac{600}{11}\right)^2$ sq. yds.

$= \frac{720000}{77} = 9351$ sq. yds. (omitting fractions).

Reducing the square yards to acres, &c., we obtain:

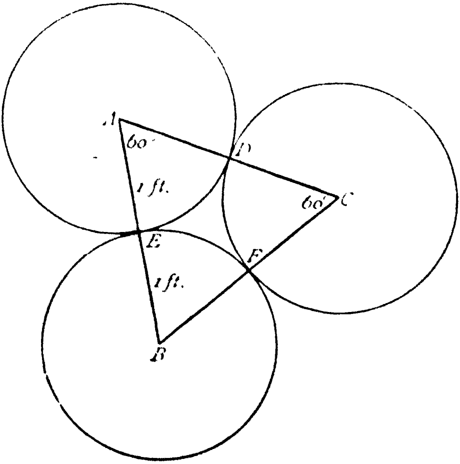
Area = 1 a. 3 r. 29 p. $3\frac{3}{4}$ sq. yds.

Since we took $\pi = \frac{22}{7}$, the number of sq. yds., which depends on the fourth figure of the answer, is not reliable. But we may say that the area to the nearest pole

= 1 a. 3 r. 29 p. Answer.

(By taking $\pi = 3{\cdot}1416$ we could, of course, find the exact number of sq. yds.)

(2) Three circles, each of radius 1 ft., touch each other externally. Find the area of the curvilinear figure included between them. ($\pi = 3{\cdot}1416$) (*Sandhurst.*)



Let A, B, C be the centres of the three equal circles, and D, E, F the points of contact. Join AB, BC, CA, which will pass through D, E, F respectively. (Euclid, iii. 12.)

Curvilinear figure DEF

$$= \text{triangle } ABC - 3 \text{ equal sectors } ADE, BEF, CDF.$$

Triangle ABC is equilateral, and its side $= 2$ ft.

$$\therefore \text{ area of } ABC = \frac{\sqrt{3}}{4} \cdot 4 = \sqrt{3} \text{ sq. ft.}$$

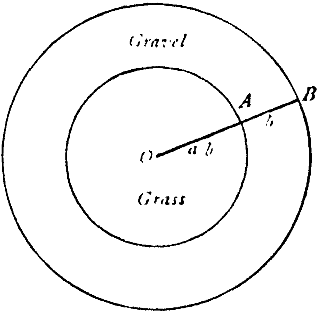
$$\text{Sector } ADE = \frac{r^2\theta}{2} = \frac{\pi}{6} \text{ sq. ft (for } r = 1 \text{ ft., and } \theta = \frac{\pi}{3}).$$

$$\therefore \text{ area required} = \sqrt{3} - \frac{\pi}{2} \text{ sq. ft.}$$

$$= 1{\cdot}73205 - 1{\cdot}5708.$$

$$= {\cdot}16125 \text{ sq. ft. Answer.}$$

(3) A circular grass-plot is surrounded by a ring of gravel b feet wide. If the radius of the circle, including the ring, be a feet, find the relation between a and b, so that the areas of grass and gravel may be equal. (*Sandhurst.*)



$$\left.\begin{array}{l} OB = a \text{ ft.} \\ AB = b \text{ ft.} \end{array}\right\}$$

Area of grass

$= \pi\,(a - b)^2$ sq. ft.

Area of gravel

$= \pi\,\{a^2 - (a - b)^2\}$ sq. ft

$$)^2 = \pi\,\{a^2 - (a - b)^2\}.$$

$$)^2 = a^2.$$

$$a - b) = a.$$

Since $a - b$ is positive, the upper sign must be taken.

$$\sqrt{2}a - \sqrt{2}b = a.$$

$$a\,(\sqrt{2} - 1) = \sqrt{2}b.$$

$$\sqrt{2} \qquad .$$

$$\therefore a : b :: 3{\cdot}4142 : 1. \quad \text{Answer.}$$

EXAMPLES ON THE AREA OF THE CIRCLE, ETC.

[N.B.—$\pi = \frac{22}{7}$, except where otherwise stated.]

(a) Area of Circle

1. Find the area of a circle whose radius is 17 ft. 7 in. Estimate also the answer correct to a sq. in., by taking

2. Find the area of a circle whose circumference is 3 ft. 8 in.

3. Find the radius in chains and links when the area is 4 a. 2 r. 35 p. ($\pi = 3.1416$.)

4. Find the diameter, if the area is 7 sq. yds. 5 sq. ft. 64 sq. in.

5. Find the circumference when the area is 3850 sq. links.

6. If the radius of one circle be to the circumference of a second as 7 to 110, and the circumference of a second to the diameter of a third as 44 to 35, show that the area of the second circle is a mean proportional between the areas of the first and third, assuming that $\pi = \frac{22}{7}$ exactly.

7. The area of a regular hexagon is half that of a regular octagon. Compare the areas of their circumscribed circles.

8. If the area of the inscribed circle of an equilateral triangle be to the area of the circumscribed circle of a regular hexagon as 4 : 9, compare the perimeters of the triangle and of the hexagon.

9. The perimeters of a regular octagon and a regular hexagon are equal. Compare the areas of their inscribed circles.

10. Find the area included between a circle whose radius is 10 in. and its inscribed regular octagon. ($\pi = 3.1416$.) Answer correct to two decimal places.

11. An equilateral triangle and a regular hexagon have the same perimeter. Show that the areas of their inscribed circles are as 4 : 9. (*Sandhurst.*)

12. Write down the expressions for (1) the circumference, (2) the area of a circle. (Radius *r*.)

The hypotenuse of a right-angled triangle is 10 ft., and one of the sides is 8 ft. Semicircles are described on the three sides of the triangle. Find the radius of the semicircle whose circumference is equal to the circumferences of the three semicircles described; and show that the area of the semicircle described on the hypotenuse is equal to the areas of the semicircles described on the two sides of the triangle. (*Sandhurst.*)

13. The perimeters of a circle, a square, and an equilateral triangle are each of them 1 ft. Find the area of each of these figures to the nearest hundredth of a sq. in. (*Sandhurst.*)

14. Find the area of a great circle of the earth, if the diameter be 7925 miles. ($\pi = 3.1416$.)

15. The rent of a circular plot of ground is £19 9*s.* 9¾*d.* at £2 5*s.* per acre. Find how many palisades, at 2 ft. interval, will be required to surround it.

16. The area of a circular table is .805 sq. in. Find how many nails, each ½ in. apart, will be required to nail on a border.

17. A circular field is surrounded by 726 yds. of fencing. How much greater is its area than that of a square field with the same perimeter? Find the rent of each at £1 2*s.* 6*d.* per acre.

18. A circular path surrounds a circular field. If the breadth of the path is 6 ft., and the circumference of the outer ring is 500 yds., find the area of the path.

19. A circular path surrounds a grass-plot 4100 sq. ft. in area. If the breadth of the path is 4½ ft., find the expense of gravelling it at 1½*d.* the sq. yd.

20. A circular path surrounds a circular flower-bed. If the outer circumference of the path is 90 ft., and the breadth of the path is 3 ft. 4 in., find to the nearest sq. yd. the number of sq. yds. in the flower-bed.

21. Find the area of a circular ring whose inner and outer radii are 8 in. and 6 in. What is the length of the ring?

22. Find the area of a circular ring, if the length is 5 ft. 4 in., and the breadth 8½ in.

23. A circular picture, whose diameter is 10 in., is set in a circular frame whose diameter is 15 in. Find the area of the space between the picture and its frame.

24. Find the length of a circular running-path, if the area of its inner circle is 26,026 sq. yds., and its breadth is 14 ft.

25. A cow is fastened by a rope 18 ft. long to a stake in the ground. What area has the cow to graze upon?

26. What is the greatest area which can be enclosed by a perimeter of 1000 yds.?

27. Find the cost of planting, at 11*d.* a sq. rod, a circular field whose circumference is 200 yds. (*Oxford Local.*)

28. A circular court has a radius of 40 ft. If a pathway be cut off from it all round the inside, and the rest turfed, find the expense of turfing it at 3½*d.* per sq. yd., given that the total area of the path is 244 sq. yds. 4 sq. ft.

29. Show how to find the radii of the n concentric circles which divide the area of a given circle into $n+1$ equal parts.
If the radius of the given circle is 1 ft., find to the nearest hundredth of an inch the radii of the two concentric circles which divide its area into 3 equal parts. (*Sandhurst.*)

30. A circle of 60 in. radius is divided by 4 concentric circles into 5 parts of equal area. What are the values of the several radii?

31. How many coins, each 1⅓ in. in diameter, can be arranged upon a circular table 20 in. in diameter, in the

form of a regular hexagon? Find the area of that portion of the table not covered by the coins,* and compare the length of a string which just passes round all the coins with the circumference of the table. (*Sandhurst.*)

32. Find the area of all the coins, the diameter of each being ¾ in., which can be arranged in the shape of an equilateral triangle, the outer side of which contains 13 coins. Find also the area of all the unoccupied spaces between the coins. ($\pi = 3.1416$.)

33. The cross-section of a tunnel is a rectangle surmounted by a semicircle. Find its area, given extreme height = 32½ ft., and breadth = 25 ft.

34. A rectangular building terminates in a semicircular apse one way. Given length (exclusive of apse) = 86 ft., and breadth = 35 ft., find the area of the ground it stands on.

35. The area of a window, which is in shape a square surmounted by a semicircle, is 24 sq. ft. Find its perimeter approximately. ($\pi = 3.1416$.)

36. One hundred yds. of palisading surround a space in the shape of a rectangle with semicircular ends, and containing 546 sq. yds. Find the width and extreme length of the enclosure, assuming $\pi = \frac{22}{7}$. (*Sandhurst.*)

37. A regular hexagon whose side is 1 ft. is divided, by drawing its diagonals, into six equilateral triangles. In each of these a circle is inscribed. Find how much of the area of the hexagon is not taken up by the circles. ($\pi = 3.1416$.)

38. A square is inscribed in a circle of known radius, a second circle in the square, a second square in the second circle, and so on. Find a series to express the areas included between each circle and the square next inscribed in it; and find the sum of all the areas which can be so formed. What does this sum become when the radius of the first circle is 1 ft.? ($\pi = 3.1416$.)

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* This seems to mean area of table *minus* area of coins.

39. In a given circle a regular hexagon is inscribed, and in the hexagon another circle, and then hexagons and circles alternately *ad infinitum.* Find a series to express the spaces included between each circle and its next inscribed hexagon, and show that the sum of all these spaces is $(4\pi - 6\sqrt{3})\,r^2$, r being the radius of the first circle.

40. In a given regular hexagon a circle is inscribed, in the circle another hexagon, and then circles and hexagons alternately *ad infinitum.* Find a series to express the spaces included between each hexagon and its next inscribed circle, and show that the sum of all these spaces is $(6\sqrt{3} - 3\pi)\,a^2$, a being the side of the hexagon.

40*a*. In Sandhurst (July, 1884) the following question was set, which combines the last two.

In a given circle a regular hexagon is inscribed, and in the hexagon another circle, and then hexagons and circles alternately *ad infinitum.* Find a series to express the spaces included between each circle and hexagon, and show that it approximates to the area of the original circle.

(b) Areas of Sector and Segment

41. The angle of a sector of a circle is 37° 40′, and the radius is 7 ft. 8 in. Find the area of the sector. ($\pi = 3.1416$.)

42. A sector of a circle whose radius is 6 ft. is equal to 24 sq. ft. 108 in. Find the length of its arc.

43. In a circle whose radius is 2 ft. find the areas of the two segments into which the circle is cut by a chord which subtends at the centre an angle of 45°. ($\pi = 3.1416$.)

44. It is required to cut out of the circle whose radius is 9 in. a sector which shall be equal to 75 sq. in. Find the number of degrees in the required sector.

45. The angle of a sector is 49° 20′, and the area of the sector is 37 sq. in. Find the area of the whole circle.

46. The length of the arc of a sector of a given circle is 16 ft., and the angle of the sector at the centre of the circle is $\frac{1}{8}$ of a right angle. Find the area of the sector; and determine also the length of the arc subtending the same angle in a circle whose radius is four times that of the given circle. (*Sandhurst.*)

47. The arc of a sector is 31.275 in., and the radius of the circle is 32 in. Find the area of the sector.

48. Given the area of a sector = 36 sq. ft., and the angle of the sector = 40°, find the arc to the nearest inch.

49. Find the area of the segment of a circle which subtends an angle 15° at the centre of a circle whose radius is 4 ft. ($\pi =$

50. Find in a circle whose radius is 6 in. the area contained between two parallel chords, which are both the same side of the centre, and subtend at it angles of 60° and 45° respectively. ($\pi = 3.1416$.)

51. If in a circle a segment 9.06 sq. in. in area subtends at the centre an angle of 60°, find the radius of the circle. ($\pi = 3.1416$.)

52. Two chords, whose lengths are 2 ft. and 7 in. respectively, are drawn at right angles from a point on the circumference of a circle. Determine the area of the circle.

53. Find the area included between a circle and two tangents which intersect at an angle of 120°. Radius of circle = 1 ft. ($\pi = 3.1416$.)

54. Two tangents drawn to a circle intersect at right angles. Show that the area included between them and the circle is $\frac{4-\pi}{4\pi}$ of the area of the circle.

55. Two equal circles, the centre of each of which is on the circumference of the other, intersect. Find the area of the space common to both of them, the radius of each being 6 in. ($\pi = 3.1416$.)

56. Two equal circles (radius of each = 2 in.) touch one another, and a common tangent is drawn. Find the space included between the two circles and their common tangent.

57. Two circles, whose radii are 10 and 30 in., touch each other externally. Find the area of the space included between the two circles and a line which touches both circles, but does not pass through the point of contact of the circles. ($\pi = 3.1416$.) (*Oxford Local.*)

58. If three equal circles, whose radii are each 7 in., touch each other, find the area included between them to three places of decimals. ($\pi = 3.1416$.) (*Sandhurst.*)

59. A window-opening consists of a rectangle surmounted by a pointed arch. The chords of the two arcs form, with the base of the arch, an equilateral triangle, and the centre of each arc is at the opposite point of the triangle. Find the height and area of the window, correct to a sq. in., if the height of the rectangle is 14 ft., and the breadth of the rectangle is 6 ft. ($\pi = 3.1416$.)

60. A circular disc of cardboard 1 ft. in diameter is divided into 6 equal sectors by pencil-lines drawn through the centre. In each sector there is described a circle touching the two bounding radii of the sector, and also the arc joining their ends at its middle point. If the circles are cut out from the 6 sectors, find the area of the cardboard remaining. (*Sandhurst.*)

CHAPTER VI.—SIMILAR FIGURES

Section I.—Lengths

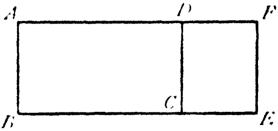
42. Similar figures are defined by Euclid as those "which have their angles equal, and their sides about their equal angles proportionals." In less exact language, we may say that similar figures have the same *shape*, but are not usually of the same *size*.

43. In the definition given by Euclid, it should be noticed that there are *two* conditions of similarity to be fulfilled:

(i) That the *angles* should be *equal*.
(ii) That the *sides* should be *proportional*.

Now it is proved in Euclid. vi. 4–7, that a triangle which has one of these two conditions has the other of necessity. If, therefore, two triangles have two angles of the one equal to two angles of the other, their third angles are equal, and consequently the triangles are similar.

But it does not follow that in *any* figure, because the angles are equal, that the sides are proportional. Take, for instance, two rectangles as in the figure. The rectangles $ABCD$, $ABEF$, have their angles equal. But they have not their sides about the equal angles proportionals; for BA is not to AD as BA is to AF, for then AD would have to be equal to AF, which is absurd.



Neither does it follow that, if the sides of two figures are proportional, their angles are equal. For example, take any square and any rhombus. They are not equiangular, but their sides are proportional.

All *regular* polygons of the same number of sides are similar figures. It follows that all circles must be regarded as similar figures.

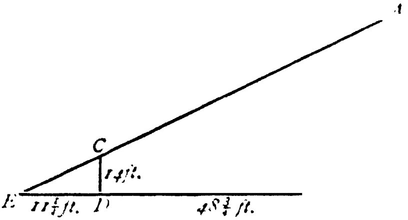
44. *The sides of similar figures are proportional.*

This is the important fact to observe. If the sides of one figure are given, and *one* of the *sides* of a figure similar to it, we can determine all the other sides by proportion. This principle we have employed once or twice already, for instance, in determining the circumference of a circle.

45. We may present similarity of figure from another slightly different point of view. Two similar figures have the same shape, but they are *not on the same scale.* The use of the last word suggests an important application of similar figures. All maps and plans of the same country and of the same object are necessarily of the same shape, but on different scales. If the scale is known, we can determine by proportion the lengths of one map from the lengths of the other, or the real lengths of an object from those in a map or plan.

Examples.—Similar triangles give us some methods for measuring heights when it is not convenient to measure angles. (See chap. I. sec. i. examples.) The first two examples will illustrate this.

(1) A person wishing to find the height of a church-steeple, observes that a lamp-post 14 ft. high is 48 ft. 9 in. distant from its base, and that from a point on the ground 11 ft 3 in. beyond the lamp-post, in the line joining their bases, the summits of the steeple and lamp-post are in the same straight line. What is the height of the steeple?



Let *AB* be the steeple, *CD* the lamp-post, *E* the point on the ground at which *C* and *A* are in a straight line.

Then the triangles EDC, EBA are similar.

$$\therefore AB : BE :: CD : DE.$$

$$AB : 48\tfrac{3}{4} + 11\tfrac{1}{4} :: 14 : 11\tfrac{1}{4}.$$

$$AB = \frac{60 \times 14}{11\tfrac{1}{4}} = \frac{\overset{4}{60} \times 14 \times 4}{\underset{3}{45}} = 74 \text{ ft. } 8 \text{ in.} \quad \text{Answer.}$$

The height of an object can also be measured by its shadow as in the following example.

(2) Find the height of a tower which casts a shadow of 100 ft. at the same moment that an upright walking-stick 3 ft. long casts a shadow of 3 ft. 9 in.

By similar triangles,

Height of tower : shadow of tower :: height of stick : shadow of stick.

Height of tower : 100 ft. :: 3 ft. : $3\tfrac{3}{4}$ ft.

$$\therefore \text{height of tower} = \frac{300}{\tfrac{15}{4}} = 80 \text{ ft.} \quad \text{Answer.}$$

(3) The scale of a map is $\frac{1}{10}$ in. to a mile. By what line on the map will a distance of 56 miles be represented? and how many miles are represented by a distance on the map of 2·375 in.?

1 mile : $\frac{1}{10}$ in. :: 56 miles : 5·6 in.

Thus 56 miles are represented by 5·6 in.

Again, $\frac{1}{10}$ in. : 1 mile :: 2·375 in. : 23·75 miles.

Thus 2·375 in. represent 23 miles 6 furlongs.

(Observe that in questions on the scale of maps and plans it is not usually necessary to reduce all the data to the same dimensions.)

EXAMPLES ON THE LENGTHS OF SIMILAR FIGURES

1. The top of a flagstaff, known to be 32 ft. high, is seen to be in line with the top of a church-spire, the common angle of elevation being 30°. If the church be 60 yds. from the flagstaff, what is the height of the spire from the ground?

2. A person standing due south of a lighthouse observes that his shadow, cast by the light at the top, is 24 ft. long. On walking 100 yds. due east, he finds his shadow to be 30 ft. Supposing him to be 6 ft. high, find the height of the light from the ground. (*Sandhurst.*)

3. A person at a known distance of 100 ft from a tower finds that, if he sets up his walking stick in the ground 1 ft. 10 in. nearer the tower, and in the same line, and then looks at it from the ground, the tops of the walking-stick and the tower are in a line. If the walking-stick is 2 ft. 9 in., find the height of the tower.

4. A ladder 22 ft. 10½ in. long leans so as to reach 18 ft. up a wall. Find the length of a ladder which, leaning at the same angle, reaches 16 ft. up the wall.

5. A, B, C, D are four points in a straight line on a horizontal plane. At A and C are flagstaffs, which subtend angles α and β respectively at B, and whose tops are in the same straight line with D. AB being 100 ft., and CD 40 ft., find BC. (*Oxford Local.*)

6. If the shadow of a man 5 ft. 10¾ in. high be 6 ft. 6 in., what is the height of a tower whose shadow at the same time is 104 ft.?

7. The shadow of a rod 3 ft. high is measured, and found to be 1 ft. 10½ in. What is the height of a flagstaff whose shadow at the same time is 25 ft.?

8. The top of a mountain is observed to be in line with a steeple whose height is known to be 75 ft. The *horizontal* distance of the mountain-top from the observer is known to be 1½ miles. If the tower is 99 ft. from the observer, find the height of the mountain.

9. DE is drawn parallel to BC, a side of the triangle ABC. If the sides of the triangle ADE are AD = 2 ft. 6 in., AE = 1 ft. 9 in., and DE = 9½ in., and also if DB = 3 ft. 4 in., find the sides of ABC.

10. $ABCD$ is a trapezoid, the four sides of which are known. The parallel sides are 32 ft. and 26 ft. The non-parallel sides AB (= 6 ft.) and DC (= 5 ft.) meet produced in E. Find the distance of E from the angular points B and C.

11. $ABCD$ is a rectangle 12 ft. by 9 ft. From B and D perpendiculars BE, DF are drawn to the diagonal AC. Find the value of EF.

12. A straight line, whose length is 11¼ in., is drawn parallel to the base of a triangle, cutting one of the sides into the segments 4 ft. 3¾ in. and 2 ft. 0¾ in. (the latter adjacent to the base). Find the base of the triangle.

13. Two similar polygons have one side of the one to the corresponding side of the other as 5 to 9. The perimeter of the first is 22 yds. 1 ft. 11 in. Find the perimeter of the second.

14. The sides of a pentagon are 5.2 in., 6.4 in., 10.7 in., 8.9 in., and 4.1 in. Find the four sides of a similar pentagon, whose side homologous to the first side of the other pentagon is 1 ft. 1 in.

15. $ABCD$ is a trapezoid, having BC and AD parallel. Given BC = 17 ft., AD = 20 ft., and distance between them 9 in., if AB and DC meet at E, find the perpendicular distance of E from AD.

16. The distance between two places is 215 miles. Find the distance in a map in which the scale is 1 inch to 20 miles; and find also the scale of a map in which the same distance will be represented by 4⅞ in.

17. The distance between two places is 47 miles, and their distance in a map is $9\frac{19}{24}$ in. On what scale is the map drawn? and what will a line of 6 in. represent on the same scale?

18. In a map, of which the scale is 1 inch to a mile, the distance between two places is 9.85 in. Find the real distance; and find also how much the same line would represent if the scale were 1 in. to a furlong.

19. In Bradshaw's railway map, in which the scale is about $\frac{1}{21}$ in. to a mile, all the lines of a certain railway come together to 1 ft. $10\frac{1}{3}$ in. Find the mileage of the railway.

20. The distance between London and Edinburgh in a map of Europe is 1 in., in a map of the British Isles 5 in. The map of the British Isles is drawn on the scale of $1\frac{1}{2}$ in. to 100 miles. Find what fraction of an inch represents a mile in the map of Europe.

21. Asia is roughly 5 times as large as Europe, and has 33,000 miles of coast-line. If Europe has 2.57 times more coast-line than Asia in proportion to its size, find the scale of a map of Europe in which the coast-line is represented by a line of 4 ft. 3 in.

22. The scale of a map of Asia is $\frac{3}{2000}$ in. to a mile, that of a map of Hindostan 1 in. to $187\frac{1}{2}$ miles. By what decimal of an inch will a line of 2.35 in. in the latter map be represented in the former? and what will be its real length?

23. Find the cost of travelling, at 1*d.* the mile, from London to Windermere, if the line of railway be represented in Bradshaw's map (scale $\frac{1}{21}$ in. to a mile) by a line of 1 ft. $0\frac{6}{7}$ in.

24. If a place *A* is on a map 6 in. due south of *C*, and another place *B* is $2\frac{1}{2}$ in. due east of *C*, while the real distance between *A* and *B* is 104 miles, find the scale on which the map is constructed.

25. A country has a length of 600 miles, and its greatest breadth is 320 miles. Find the dimensions of the paper on which a map of the country might be drawn, the scale being $\frac{1}{2}$ in. to the linear mile. The road from London to Bath (108 miles) is marked in a map by a line of 12 in. in length. To what scale is the map drawn? (*Sandhurst.*)

Section II.—Areas

46. *The areas of similar figures are proportional to the squares of their corresponding sides.*

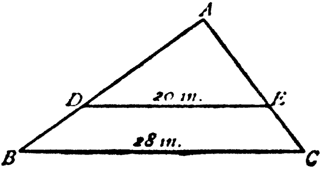
This important proposition is proved by Euclid in vi. 19, 20. In vi. 19 he proves that "similar triangles are in the duplicate ratio of their homologous sides." This, translated into simpler language, means that similar triangles are in the ratio of the squares of their sides which correspond.

In vi. 20 Euclid extends this proposition to polygons, proving that similar polygons can be divided into the same number of similar triangles, and as a consequence that the polygons also are in the duplicate ratio of their homologous sides.

47. In all cases the areas of similar figures are proportional to the squares of corresponding lengths other than the sides. For example, the areas of similar triangles are proportional to the squares of their altitudes, as the student can easily show. Again, similar regular polygons are proportional to the radii of their inscribed and circumscribed circles. Finally, circles are proportional to the square of their radii, as it has been already shown.

48. The most important application, as before, is when areas of figures have to be determined from those of similar figures in a map or plan, the scale of which is known.

Examples.—(1) The sides of a triangle are 1 ft. 5 in., 2 ft. 1 in., and 2 ft. 4 in. A straight line is drawn parallel to the longest side, and terminated by the other two sides. If the length of this straight line is 1 ft. 8 in., find the areas of the two parts into which it divides the triangle.



The area of the triangle ABC will be found $=210$ sq. in.

$\therefore$ since the triangles ADE, ABC are similar,

Area of ADE : area of ABC :: DE^2 : BC^2.

Area of ADE : 210 sq. in. :: 20^2 : 28^2.

$$\text{Area of } ADE=\frac{\overset{30}{\cancel{210}}\times\overset{5}{\cancel{20}}\times\overset{5}{\cancel{20}}}{\underset{4}{\cancel{28}}\times\underset{7}{\cancel{28}}}=\frac{750}{7}=109\tfrac{1}{7}\text{ sq. in.}$$

$\therefore$ area of $DBCE=210-109\frac{1}{7}=100\frac{6}{7}$ sq. in. Answer.

(2) The scale of one map is 3 in. to 20 miles, that of another is 2 in. to 30 miles. What area in the second will correspond to an area of 6·75 sq. in. in the first? and how many sq. miles will it represent?

On the first map, 3 in. represent 20 miles.
$\therefore$ 9 sq. in. „ 400 sq. miles.
$\frac{9}{400}$ sq. in. „ 1 sq. mile.
On the second map, 2 in. represent 30 miles.
$\therefore$ 4 sq. in. „ 900 sq. miles.
$\frac{4}{900}$ sq. in. „ 1 sq. mile.

Thus $\frac{9}{400}$ sq. in. in the first map corresponds to $\frac{4}{900}$ sq. in. in the second.

$$\therefore \frac{9}{400} : \frac{4}{900} :: \overset{\text{sq. in.}}{6{\cdot}75} : \text{answer}.$$

$$81 : 16 :: 6\tfrac{3}{4} : \frac{27\times\overset{4}{\cancel{16}}}{\cancel{4}\times\underset{3}{\cancel{81}}}=\tfrac{4}{3}.$$

Answer $1\frac{1}{3}$ sq. in.

$$\text{Again,}\quad \overset{\text{sq. in.}}{9} : \overset{\text{sq. miles.}}{400} :: \overset{\text{sq. in.}}{6\tfrac{3}{4}} : \text{number of sq. miles.}$$

$$\therefore \text{number of sq. miles represented}=\frac{400\times27}{4\times9}=300.$$

Answer 300 sq. miles.

(3) What fraction of their natural size will areas be represented on a map whose scale is 11 in. to the mile?

11 in. represent a mile.
$\therefore$ 121 sq. in. „ a sq. mile.

$$\frac{\cancel{121}}{144\times9\times\underset{40}{\cancel{1760}}\times640}\text{ sq. mile represent a sq. mile.}$$

Thus a sq. mile is represented by the above fraction of itself,

which $=\dfrac{1}{33{,}177{,}600}$. Answer.

EXAMPLES ON THE AREAS OF SIMILAR FIGURES

1. Prove that two similar triangles are to one another as the squares of their altitudes.

2. Prove that two regular polygons of the same number of sides are to one another as the squares of the radii of their inscribed or circumscribed circles.

3. The sides of a triangular field are 300, 400, and 500 yds. If a belt 50 yds. wide is cut off the field, what are the sides of the interior triangle? and what is the area of the belt? (*Sandhurst.*)

4. The cross-section of a leaden pipe is a regular hexagon, the side of which is ⅛ in. If the perimeter of the pipe is increased to 3 in., in what proportion will the area of the bore be increased?

5. The side of a square field is 5 times as long as that of a second. If 16 bushels of potatoes are produced by the latter, how many bushels may the former be expected to produce under equal conditions?

6. The areas of two circular ponds are to one another as 25 to 36. If it costs £17 4*s.* 8*d.* to surround the former with a stone rim, what will it cost to surround the latter?

7. Find the area of a field, in shape a right-angled triangle, given that the perpendicular from the right angle on the base is to one of the segments of the base as 5 to 2, and the side adjacent to the other segment of the base is 15 chains.

8. A corner of a triangular field is cut off by a straight line parallel to one side. If this line is 50 yds., and the field is divided into two parts which are as 1 : 8, the part cut off being the smaller, determine the side of the field opposite to the corner cut off.

9. The sides of a right-angled triangle are 15 in. and 20 in. On the side equal to 15 in. a triangle is drawn whose other two sides are 13 in. and 14 in., and similar triangles on the other two sides of the right-angled triangle. Prove by calculation that the area of the triangle drawn on the hypotenuse = the sum of the areas of the other two triangles.

10. A rectangular field is 160 yds. long, and 70 yds. broad. A similar rectangular enclosure is built in one corner, the longer side of it being 30 ft. Find its area.

11. In a known triangle, show how to find the distances from the vertex of the $n-1$ straight lines parallel to the base which divide the triangle into n equal parts. Example: The sides of a triangle are 10 in., 10 in., and 1 ft. Find how the equal sides are divided by lines parallel to the base, dividing the triangle into 3 equal parts.

12. AC, the diagonal of a rectangle $ABCD$, is divided into 4 parts such that the perpendiculars dropped from each on to the sides AB, AD divide the rectangle into 4 equal parts. Find the proportion in which the 4 segments of the diagonal are to one another.

13. The sides of two similar quadrilateral fields are to each other as 3 : 4. The expense of enclosing the first is £28 17*s.*, and the rent is £6 13*s.* 6*d.* Find the cost of enclosure and rent of the second.

14. A circle has its diameter equal to the circumference of a second circle. The area included between the first circle and its inscribed regular heptagon is 320 sq. in. Find to two places of decimals the area included between the second circle and its inscribed regular heptagon.

15. The equal sides of an isosceles triangle, AB, AC, are each 5 in., and the base is 6 in. D and E are points in the sides AB and AC, such that $AD = 3$ in., and $AE =$ 2 in. From D and E, DF, EG are drawn perpendicular to the base. Find the area of the pentagon $ADFGE$.

16. The Ordnance Survey has a scale of 25 in. to the mile. What fraction of their natural size will objects be represented?

17. A ground-plan of a house is $\frac{1}{14400}$ of the size of the house. On what scale is it drawn? and what will the real area of a room be whose length and breadth on the plan are 2.75 in. and 1.32 in.?

18. A photograph is enlarged to 3 times its original size. The original breadth was $2\frac{1}{3}$ in. What is the enlarged breadth? The enlarged length is 7 in. What was the original length?

19. In a map of England, which is on a scale of 3 in. to 100 miles, the area of a certain county is approximately 1.08 sq. in. Find (1) its real area; (2) its area on a map where objects are the $\frac{1}{1,200,000,000,000}$ part of their natural size.

20. A microscope magnifies objects 250 times. If the apparent perimeter and area of an object under the microscope are 3 in. and 6 sq. in., find its real perimeter and area.

21. The dimensions of a quadrilateral field $ABCD$, which has a right angle at A, are thus laid down in a plan constructed on a scale of 1 in. to 400 yds.: $AB = .45$ in., $BC = .65$ in., $CD = .7$ in., $DA = .6$ in. Find the area of the field in acres and yards.

22. The lengths of the sides of a triangular field are 100 yds., 170 yds., and 210 yds. What will be its area on an Ordnance Survey map of 25 in. to the mile?

23. Find the area of the triangular field ABC from the following measurements on the Ordnance Survey of 25 in. to the mile: $AC = 4.1$ in., perpendicular from B on $AC =$ 1.59 in. Calculate the same area from the three sides, AB measuring 3.3 in., and BC 2 in. Express the mean of the two in acres. (*Sandhurst.*)

24. One map is constructed on a scale of 25 miles to an inch, another of 25 inches to a mile. If the second map is 5 times as large as the first, compare the areas represented by them. By what area in each would an extent of 1000 acres be represented?

25. If on a map a square piece of ground 500 sq. miles in extent is represented by $\frac{4}{5}$ sq. in., what is the length of a river which is represented by a line of 4.4 in.?

26. Find the area of a rectangular room, the length and breadth of which are 14 ft. 8 in. and 9 ft. 3 in. on a ground-plan whose scale is 1 in. to 8 ft. Answer to two places of decimals.

27. The area of a quadrilateral field, whose diagonals intersect at right angles, is 3 a. 3 r. On a plan of the estate of which the field is a part, the diagonals are represented by lines $8\frac{4}{5}$ in. and $6\frac{2}{3}$ in. in length. Find how many linear inches in the plan represent a mile.

28. Two maps are of the same size. On the first a line 8.56 in. in length represents 128.4 miles; on the second an area of 100 acres is represented by $\frac{1}{480}$ sq. in. Compare the areas represented by the two maps.

29. The homologous sides of two similar triangles are to each other as 13 : 29. Find to three places of decimals how many times the second is larger than the first.

30. The areas of two regular polygons of the same number of sides are 155.48 sq. in. and 271.36 sq. in. If the radius of the circumscribed circle of the first be 2 ft. 6 in., find the radius of the circumscribed circle of the second, to three decimal places of an inch.

Book II.

CHAPTER I.

PLANES, SOLID ANGLES AND FIGURES

49. So far we have confined ourselves to plane figures. We now proceed to consider *solids* or *Solid Figures.*

DEFINITION 1.—"*A solid is that which has length, breadth, and thickness.*" (Euclid, xi. def. 1.*)

Plane figures have *two* dimensions, length and breadth; solid figures, *three*, length, breadth, and thickness.

DEFINITION 2.—"*That which bounds a solid is a superficies.*" (Euclid, xi. def. 2.)

That is, a *surface.* This surface is either curved or plane. A plane surface, or simply a *plane*, is a surface "in which, any two points being taken, the straight line which joins them lies wholly in that surface." (Euclid, i. def. 7.)

The walls and ceiling of a room are planes or plane surfaces. In plane figures all the lines containing them lie in the same plane. In treating of plane figures we have not always supposed all the figures themselves to be in the same plane. For instance, we measured the walls of a room, which are rectangles in different planes; and also in measuring heights and distances we had to do with triangles not always in one and the same plane.

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* The definitions in this part will be mainly drawn from Euclid, book xi. But no knowledge of that book is required for a student to understand thoroughly what follows.

To understand solid figures, we must consider a little the relations of planes to each other, and to lines which meet them, not being in the same plane.

50. *Planes.*

(1) Through three points, not in the same straight line, one plane, and one only, can be drawn.

Also through two straight lines which cut one another only one plane can be drawn.

(2) If two planes cut one another, their common section is a straight line.

Thus the plane of a wall of a room meets the plane of the ceiling in a straight line.

(3) DEFINITION 3.—"*A straight line is perpendicular, or at right angles to a plane, when it makes right angles with every straight line meeting it in that plane.*"

For example, a pole fixed vertically upright will be at right angles to the plane of the ground, and consequently make right angles with any horizontal line. This result has been already assumed in the measurement of heights and distances. (See page 9, example 2.)

If two straight lines are parallel, and one is perpendicular to a plane, the other will be also perpendicular to that plane; and conversely two perpendiculars to the same plane will be parallel.

(4) DEFINITION 4.—"*A plane is perpendicular to a plane when the straight lines drawn in one of the planes perpendicular to the common section of the planes are perpendicular to the other.*" (Euclid, xi. def. 4.)

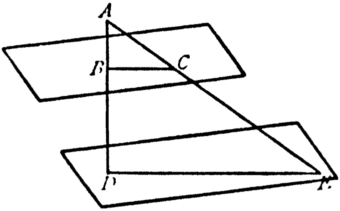
The student will readily see that the side-wall of a room is perpendicular to the floor.

(5) DEFINITION 5.—"*Parallel planes are such as do not meet if produced.*" (Euclid, xi. def. 8.)

Thus the floor of a room is parallel to the ceiling.

If two planes are parallel, a straight line perpendicular to one will be perpendicular to the other.

(6) Also if two parallel planes are cut by a third plane, their sections are parallel.

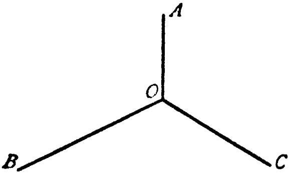


Let the two straight lines AB, AC be cut by two parallel planes. Then their sections BC, DE are parallel. Hence $AB : BD :: AC : CE$ (Euclid, vi. 2), and also [illegible] are similar.

These results hold, whatever be the number of cutting parallel planes.

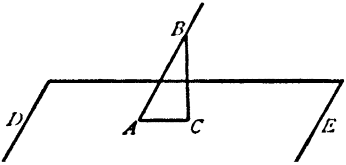
51. *Solid angles.*

Definition 6.—"*A solid angle is that which is made by more than two plane angles, which are not in the same plane, meeting at one point.*"



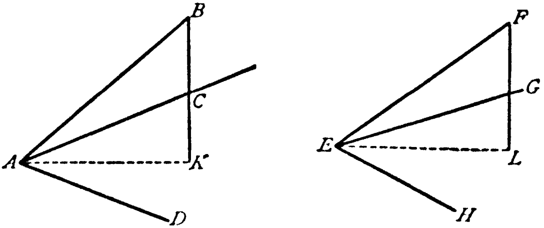
For instance, if the three angles AOB, AOC, BOC, all in different planes, meet at the point O, they form a solid angle. At every corner of a room there is a solid angle, formed by three right angles meeting.

52. *Inclination of a straight line to a plane.*



The inclination of the straight line AB to the plane DCE is measured as follows: Take any point B in AB, and draw BC perpendicular to the plane. Join AC. Then BAC is called the inclination of AB to the plane.

If two solid angles at A and E are contained by equal angles, the solid angles themselves will be equal. Also the inclination of AB to the plane in which are AC and AD will be equal to the inclination of EF to the plane in which are EG and EH. This may be proved by superposition.



The results of this section will not be required until the chapter on Similar Solids.

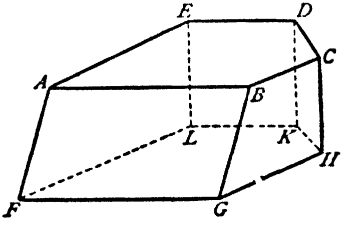
53. *Polyhedra.*

The solid figures we are about to treat of may be divided into two classes: (1) Polyhedra.
(2) Solids of revolution.

DEFINITION 7.—*A polyhedron is a solid figure bounded on all sides by planes.*

These planes intersect in straight lines, which are called the *edges* of the polyhedron; the plane figures they form are called the *faces* of the polyhedron; and at every point where more than two planes meet there is a *solid angle.*

Hence we may say that a polyhedron is a solid figure *contained by plane figures.*



For example, in the accompanying diagram we have a polyhedron contained by seven plane *faces*, two pentagons and five quadrilaterals; there are fifteen edges, and ten solid angles.

A polyhedron is *regular* when all its angles are equal, and all its faces equal and regular polygons of the same number of sides.

The only polyhedra we shall treat of are:

(i) The *prism*, with its particular cases, the *parallelepiped* and the *cube*.

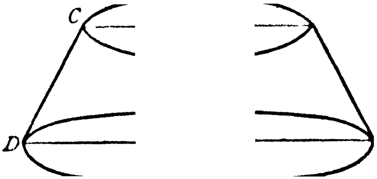
(ii) The *pyramid*, with its particular case, the *tetrahedron*.

54. *Solids of revolution.*

Definition 8.—*A solid of revolution is the figure described by the revolution of a plane figure round a fixed straight line.*

The fixed straight line is called the *axis*. In the cases we shall treat of, the axis will be usually one of the sides of the revolving figure.

For instance, let the side AB of the trapezium $ABDC$ remain fixed, and the trapezium be made to revolve round it. Then a solid figure is traced out by other three sides revolving, to which we shall afterward give the name *frustum of a cone*.



The solids of revolution we shall treat of are:

(i) The *cylinder*.
(ii) The *cone*.
(iii) The *sphere*.

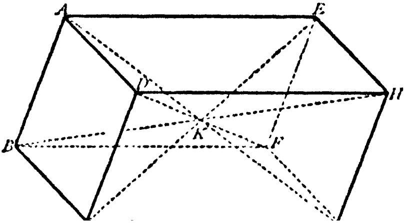
55. *A frustum of a solid* is the part of it which is cut off by a plane; but as the planes will be differently drawn in different cases, the frusta will be specially defined for each figure.

CHAPTER II.—THE PARALLELEPIPED

Section I.

56. DEFINITION 9.—*A parallelepiped is a solid figure contained by six parallelograms, of which every opposite two are equal and parallel.*

For instance, the annexed figure represents a parallelepiped, contained by six parallelograms, of which every opposite two are equal; namely, $ABCD$, $EFGH$; $ADHE$, $BCGF$; $ABFE$, $DCGH$. These six parallelograms are called the *faces* of the figure. Any one of them may be taken for *base*, and the four adjacent to the base will be called the side faces.



A parallelepiped has twelve edges, and eight solid angles. The edges divide themselves into three groups, each consisting of four equal and parallel edges. Thus in the figure AB, DC, EF, GH are all equal; and AD, BC, EH, FG are all equal; and AE, DH, CG, BF are all equal. Hence we talk of the *three edges* of a parallelepiped. They are the same as the *three dimensions* of a solid body, and may be considered its length, breadth, and thickness. Each of the solid angles is contained by three plane angles, and it will easily be seen that if the angles at one point are known, the angles at any other point can be at once determined, since they are either equal or supplementary to the former. Thus the angles at C are BCD, DCG, and BCG;

and of the angles at F, $BFG = BCG$, and EFG, EFB are supplementary to DCB and DCG respectively. Thus it is usual to talk of the *three angles* of a parallelepiped, meaning the three plane angles which unite to form the solid angle at any point.

N.B.—It will be shown in the next chapter *that the parallelepiped is a particular case of the prism*, and the student must carefully bear this fact in mind, and not be misled, because the former, in consequence of its great practical importance, is discussed separately.

57. A *diagonal* of a parallelepiped is a straight line joining two opposite corners. Every parallelepiped has therefore four diagonals; i.e. AG, BH, CE, and DF in the figure. All these diagonals meet at the same point K, and are bisected at it.

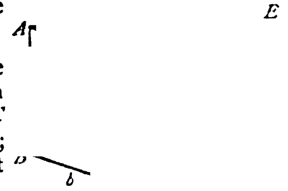
The *altitude* of a parallelepiped is the perpendicular drawn to the base from any point of the opposite face. It must be remembered here that *any* side may be taken for base.

58. *Rectangular Parallelepiped. Cube.*

Parallelepipeds are divided into rectangular and oblique-angled. If the six parallelograms are *rectangles*, the parallelepiped is called *rectangular*. This is the simplest and most familiar shape a solid figure can take. A common brick, an ordinary square box or cistern, a regularly-built four-sided room, are all excellent examples of a rectangular parallelepiped. It is clear that each of the edges is at right angles to the two faces that meet it. Since all the angles are right angles, a rectangular parallelepiped is completely determined when its three edges are known. Thus a brick is completely determined when we know its *three dimensions*, its *length, breadth, and thickness.*

If the three edges are all equal, the figure is called a *cube.* Thus a cube is a rectangular parallelepiped whose length, breadth, and thickness are the same. The faces of a cube are obviously six equal squares. Since all the edges are equal, it is usual to speak of *the edge of a cube.*

59. In a rectangular parallelepiped all the diagonals are equal, and may be simply expressed.



Let a, b, c be the three dimensions. Join CE, CF. Then EFC will be a right angle; for EF is at right angles to the base.

$$CE^2 = CF^2 + FE^2 = CG^2 +$$

$$\therefore \text{ diagonal} = \sqrt{a^2 + b^2 + c^2}.$$

In a cube $a = b = c$.

$$\therefore \text{ diagonal of cube} = \sqrt{3} \,.\, a.$$

60. The exterior surface of a parallelepiped is the sum of six parallelograms.

The exterior surface of a rectangular parallelepiped is the sum of six rectangles, which in the case of the cube are squares.

Hence in the rectangular parallelepiped, if a, b, and c are the three dimensions,

$$\text{Exterior surface} = 2\,(ab + ac + bc).$$

And in the cube, if edge $= a$,

$$\text{Exterior surface} = 6a^2.$$

Examples.—(1) The diagonal of a cube is 2 ft. 4 in. Find the exterior surface of a second cube whose diagonal is equal to the edge of the first cube.

$$\text{Edge of first cube} = \frac{1}{\sqrt{3}} \text{ diagonal} = \frac{28}{\sqrt{3}} \text{ in.}$$

$$\therefore \text{ diagonal of second cube} = \frac{28}{\sqrt{3}} \text{ in.}$$

$$\text{Edge of second cube} = \frac{1}{\sqrt{3}} \,.\, \frac{28}{\sqrt{3}} = \tfrac{28}{3} \text{ in.}$$

$$\therefore \text{ exterior surface of second cube}$$

$$= 6\left(\tfrac{28}{3}\right)^2 = \tfrac{6}{9} \,.\, 784 = \tfrac{2}{3} \times 784 \overset{\text{sq in.}}{}$$

$$= 3 \text{ sq. ft. } 90\tfrac{2}{3} \text{ sq. in.} \quad \text{Answer.}$$

(2) The dimensions of a rectangular parallelepiped are as 21 : 16 : 12, and the diagonal is 87 ft. Find the exterior surface.

Let the three dimensions be $21x$, $16x$, and $12x$.

$$\therefore \sqrt{441x^2 + 256x^2 + 144x^2} = 87 \text{ ft.}$$
$$\sqrt{841x^2} = 87.$$
$$29x = 87.$$
$$x = 3.$$

Again, exterior surface $= 2\,(21x \times 16x + 21x \times 12x + 16x \times 12x)$.

$$= 1560x^2 = 1560 \times 9 \text{ sq. ft.}$$
$$= 1560 \text{ sq. yds.} \quad \text{Answer.}$$

(3) The three edges of a parallelepiped which meet at a point B (see figure, p. 110) are $BA = 10$ ft., $BC = 8$ ft., $BF = 14$ ft., and the angles are $ABC = 120°$, $ABF = 45°$, $CBF = 90°$. Find the exterior surface correct to two decimal places of a sq. ft.

Surface

$$= 2AB \,.\, BC \sin ABC + 2AB \,.\, BF \sin ABF + 2BC \,.\, BF \sin CBF.$$
$$= 2 \times 10 \times 8 \sin 120° + 2 \times 10 \times 14 \sin 45° + 2 \times 8 \times 14 \sin 90°.$$
$$= 160 \,.\, \frac{\sqrt{3}}{2} + 280 \,.\, \frac{1}{\sqrt{2}} + 224 \text{ sq. ft.}$$
$$= 80\sqrt{3} + 140\sqrt{2} + 224.$$
$$= 80 \times 1{\cdot}732 + 140 \times 1{\cdot}4142 + 224.$$
$$= 138{\cdot}56 + 197{\cdot}988 + 224.$$
$$= 560{\cdot}548. \qquad \text{Answer } 560{\cdot}55 \text{ sq. ft.}$$

(4) Find the cost of varnishing the outside of a plain deal box 2 ft. 6 in. long, 1 ft. 6 in. broad, and 1 ft. 3 in. high, at $4\frac{1}{2}d.$ per sq. ft.

Exterior surface $= 2\,(2\frac{1}{2} \times 1\frac{1}{2} + 2\frac{1}{2} \times 1\frac{1}{4} + 1\frac{1}{2} \times 1\frac{1}{4})$ sq. ft.

$$= 2\left(\tfrac{15}{4} + \tfrac{25}{8} + \tfrac{15}{8}\right) = \tfrac{35}{2} \text{ sq. ft.}$$

$\therefore$ cost of varnishing $= \frac{35}{2} \times \frac{9}{2}d. = \frac{315}{4}d. = 6s.\ 6\frac{3}{4}d.$ Answer.

EXAMPLES ON THE PARALLELEPIPED

(LENGTHS AND SURFACES)

1. Find the diagonal of a cube whose edge is 42 ft. 7½ in. correct to the nearest inch.

2. The diagonal of a cube is 25 ft. 3 in. Find its edge correct to the nearest inch.

3. Prove that the total exterior surface of a cube is twice the square on the diagonal. Find the total exterior surface of a cube whose diagonal is 4 yds. 1 ft. 10 in.

4. The edge of a cube is 6 in. Find the exterior surface of a cube, the edge of which is equal to the diagonal of the first cube.

5. The surface of a cube is 120 sq. yds. 6 in. Find its edge.

6. Find the diagonal of a rectangular parallelepiped whose dimensions are 48 ft., 54 ft., and 72 ft.

7. The diagonal of a rectangular parallelepiped is 30 ft.; the length and breadth of the base are each 20 ft. Find the height.

8. Find the total surface of a rectangular parallelepiped whose dimensions are 23 ft. × 16 ft. × 10 ft. 6 in.

9. In any rectangular parallelepiped, if the sum of the three dimensions be 13 ft., and the diagonal be 9 ft., find the total exterior surface. If the sum of three dimensions be s, and diagonal d, then surface $= s^2 - d^2$.

10. The dimensions of a rectangular parallelepiped are 12 ft., 7 ft., and 9 ft. Find the edge of a cube having an equal exterior surface.

11. A rectangular parallelepiped, whose altitude is 10 in., stands on a square base. Find the side of the base, so that the parallelepiped may have its surface equal to that of the cube whose edge is 40 in.

12. In a rectangular parallelepiped the three dimensions are to each other as 3 : 4 : 7, and the total surface is 1098 sq. yds. Find the dimensions.

13. If the dimensions of a rectangular parallelepiped are a, b, c, prove that the area of the three plane sections drawn through two opposite edges are $c\sqrt{a^2+b^2}$, $a\sqrt{b^2+c^2}$, and $b\sqrt{a^2+c^2}$.

14. If 3 in., 4 in., and $5\frac{1}{4}$ in. are the dimensions of a rectangular parallelepiped, find the edge of a cube whose exterior surface is twice the sum of the three plane sections drawn through the opposite edges of the parallelepiped.

15. $ABCD$ is the base of a rectangular parallelepiped; $EFGH$ being the upper surface. Through HD a plane is drawn at right angles to the base, cutting the face $BCGF$ at KL. Supposing the part cut off by the plane is detached, find how much smaller the surface of the resulting solid will be than that of the parallelepiped. ($AB = 4$ ft., $BC = 9$ ft., [illegible] $= 5$ ft., and $BK = 6$ ft.)

16. In an oblique parallelepiped the three sides meeting at a point A are $AB = 6$ in., $AC = 8$ in., $AD = 4$ in. The three angles at the same point are $BAC = 135°$, $CAD = 120°$, $DAB = 60°$. Find the exterior surface.

17. The interior length and breadth of a rectangular box are 3 ft. and 6 ft. If a stick 7 ft. in length rest cross-wise in the box, how far from the bottom of the box will the upper end be?

18. How many square feet of metal will be required to construct a rectangular tank (open at the top) 12 ft. long, 10 ft. broad, and 8 ft. deep? (*Sandhurst.*)

19. How many square feet of metal will be required to make an open cubical tank, the height of which is 2 ft. 4 in.?

20. An open square box 9 in. long, and 1½ in. deep, is made out of a piece of cardboard 1 ft. sq. How much of the cardboard will be wasted? (*Sandhurst.*)

21. Find the cost of painting the outside of a rectangular box 1 ft. × 10 in. × 8in. at 1*s.* 10½*d.* per sq. ft.

22. A drawer is 2 ft. long, 1 ft. 3 in. wide, and 4 in. deep (interior dimensions). Find the cost of lining its whole inside surface with green baize 45 in. wide, at 1*s.* 6*d.* per yd.

23. The expense of painting a cubical box was 9*s.* 6¾*d.*, at 8½*d.* per sq. ft. Find the length of its edge.

24. Find the exterior surface of a cubical box which can just pass through a door 2 ft. 9 in. wide, leaving ½ in. to spare.

25. A cube of marble, of which an edge is 1 ft., has all its corners evenly ground down so as to leave facets in the shape of equilateral triangles, while the faces of the original cube again become squares. Find approximately the total area of the body so formed. (*Sandhurst.*)

Section II.—Volumes

[FORMULÆ:

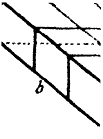
(1) Volume of *rectangular parallelepiped* = abc.
(2) „ *cube* (edge a) = a^3.
(3) „ *any parallelepiped* = base × altitude.
(4) „ *oblique-angled parallelepiped* = right section × length.]

61. It is not an integral part of the scheme of this work to prove the formulæ in Solid Mensuration. Only such explanations and illustrations as will show the reasonableness of the formulæ are given, together with those proofs which are of a simpler order, such as the one in the next section.

62. *Rectangular parallelepiped* (*volume* = abc).

We must first settle on a unit of volume. If the inch is the unit of linear measure, the cubic inch will be the unit of volume. A cubic inch is a cube whose edge = 1 inch; so that length, breadth, and height, all = 1 inch.

Let the length be a inches, the breadth b inches, and height c inches. Then area of base = length × breadth = ab square inches. Divide the base into its square inches, and draw a plane parallel to the base, at a distance from it of 1 inch. This plane will cut off a slice of the parallelepiped containing ab cubic inches; and since there are c inches in the height, the whole volume can be divided into c such slices, each containing ab cubic inches.



Hence volume of parallelepiped

$= abc$ cubic inches . . (1). Q.E.D.

For instance, if a rectangular parallelepiped has for its dimensions 2 in., 3 in., and 5 in., its volume $= 2 \times 3 \times 5 =$ 30 cubic inches.

Since base = length × breadth, another way of expressing this formula is: Volume = base × height.

The reasoning in the above proof should be carefully noticed, as it is the foundation of all theorems on the volumes of solids. The student will observe its analogy with the proof of the formula for the area of a rectangle in bk. i. chap. ii. The proof will obviously be unaffected, whatever unit of measurement be taken.

Examples.—The rectangular parallelepiped has a greater variety of practical applications than any other figure. Of these only a few can be illustrated here.

(1) How many superficial feet of inch plank can be sawn out of a log of timber 20 ft. 7 in. long, 1 ft. 10 in. wide, and 1 ft. 8 in. deep? (*Sandhurst.*)

$$\text{Volume of the timber} = \overset{\text{cub. in.}}{247 \times 22 \times 20} = \overset{\text{cub. in}}{108{,}680}.$$

Now if the timber were sawn into planks of equal length 1 in. thick, and these placed side by side, the whole would form a rectangular parallelepiped 1 in. thick, and 108,680 cub. in. in volume.

$$\therefore \text{ area of top surface} = \overset{\text{sq. in.}}{108{,}680} = 754 \text{ sq. ft. } 104 \text{ sq. in. Answer.}$$

(Note that the word 'superficial' must be understood from the context to refer only to the top surface of the wood, not to the whole surface.)

(2) Find the number of bricks, $9 \times 4\frac{1}{2} \times 3$ in. in dimensions, required to build a wall 96 yds. long, 7 ft. 6 in. high, and 9 in. thick; and their cost at 2*s.* 1*d.* per 100.

$$\text{Volume of wall} = 96 \times 3 \times 12 \times 90 \times 9 \text{ cubic in.}$$

$$\therefore \text{ number of bricks required} = \frac{96 \times \cancel{3} \times 12 \times \overset{20}{\cancel{90}} \times \cancel{9}}{\cancel{9} \times \cancel{4}{\cdot}5 \times \cancel{3}}.$$

$$= 96 \times 240 = 23{,}040.$$

$$\text{Cost of bricks} = \frac{\cancel{25}}{\cancel{100}} \times \overset{5760}{\cancel{23040}} = 5760d. = £24.$$

(3) A rectangular reservoir is 110 ft. long by 64 ft. broad. At what rate of speed per hour must water flow into it, through a pipe whose cross-section is a square (side = 2 in.), in order to make the water rise 2 ft. in 8 hours? Find also to the nearest gallon how many gallons are poured in per hour, if each gallon contains $277\frac{1}{4}$ cubic inches.

Amount of water poured into reservoir $= 110 \times 64 \times 2$ cubic ft.

Amount poured in in one hour $= \frac{110 \times 64 \times 2}{8}$ cubic ft.

$\therefore$ $\frac{110 \times 64 \times 2}{8}$ cubic ft. flow through the pipe per hour.

And cross-section of pipe $= (\frac{1}{6})^2 = \frac{1}{36}$ sq. ft.

$\therefore$ rate in feet that water flows through tube $= \frac{110 \times 64 \times 2 \times 36}{8}$.

$\therefore$ rate in miles that water flows through tube

$$= \frac{\cancel{110} \times \overset{\cancel{4}}{\cancel{64}} \times \cancel{2} \times \overset{12}{\cancel{36}}}{\underset{\cancel{2}}{\cancel{8}} \times 3 \times \underset{\cancel{16}}{\cancel{1760}}} = 12 \text{ miles.}$$

Answer 12 miles per hour.

Again, number of cubic feet poured in each hour

$$= \frac{110 \times 64 \times 2}{8} = 110 \times 8 \times 2.$$

$\therefore$ number of cubic inches poured in $= 110 \times 8 \times 2 \times 1728$.

" gallons " $= \frac{110 \times 8 \times 2 \times 1728}{277\frac{1}{4}}$.

$$= \frac{1760 \times 1728 \times 4}{1109}.$$

This result, when worked out, gives the answer 10,969 gallons (to the nearest gallon).

(Several points in this problem are worth notice. First, since water always keeps a level surface, the water poured in will assume the shape of a parallelepiped, the third dimension being given by its depth. Secondly, the water while in the pipe is also in shape a rectangular parallelepiped, two of the dimensions being given by the cross-section, and the *third by the rate the water flows;* for obviously water which flows through a pipe at the rate of ten miles an hour can be regarded as filling a pipe ten miles long.)

(4) *Duodecimal Multiplication.*—The labour of multiplying three dimensions together, when each is expressed in feet and inches, may be shortened, as in the following example.

Find the volume of a room whose dimensions are 14 ft. 2 in., 11 ft. 5 in., and 7 ft. 8 in.

ft.	in.		
14	2		
11	5		
155	10		
5	10	10	
161	8	10	
7	8		
1132	1	10	
107	9	10	8
1239	11	8	8

The result of multiplying 14 ft. 2 in. by 11 ft. 5 in. may be read 161 sq. ft. 8 primes 10 sq. in. (See bk. i. pp. 34, 35.) It will be remembered that a superficial prime = 12 sq. in. After this the work proceeds according to the following rule: Place the third dimension under the product of the other two; multiply first by the feet, then by the inches, beginning one place further to the right, and add up the two lines. *In every case carry at 12.*

Then the first number in the result is got by multiplying sq. ft. by feet, and therefore represents cubic feet.

The second is got by multiplying feet by primes, or sq. ft. by inches. Thus every unit represents 144 cubic inches. These are usually called (*cubical*) *primes.*

The third is got by multiplying sq. inches by feet, or primes by inches. Thus every unit represents 12 cubic inches. These are usually called (*cubical*) *seconds.*

The last is got by multiplying sq. in. by inches, and thus represents cubic inches.

Thus the above result = 1239 cubic feet 11 primes 8 seconds 8 cubic inches. To reduce the primes and seconds to cubic inches, multiply twice by 12, adding in seconds and cubic inches as in ordinary reduction. Thus 11 primes 8 seconds = $11 \times 12 + 8 = 140$ seconds, and 140 seconds 8 cubic inches = $140 \times 12 + 8 = 1688$ cubic inches.

Therefore volume = 1239 cub. ft. 1688 cub. in. Answer.

63. *Cube* (volume $= a^3$).

The cube is a particular case of the rectangular parallelepiped, where all the three dimensions are equal. In the language of the last section, the base will contain a^2 sq. inches, so that the volume will consist of a rows of a^2 cubic inches $= a^3$ cubic inches . . . (2).

Examples.—(1) Find the weight of a cubical block of marble, whose edge is 2 ft. 7 in., if 1 cubic foot of marble weighs 2716 oz.

Volume of marble $= (2 \text{ ft. } 7 \text{ in.})^3 = 31^3 = 29791$ cub. in.

$$\therefore \text{ weight of marble} = \frac{2716 \times 29791}{1728} \text{ oz.}$$

This answer, if worked out to the nearest ounce, becomes 46824 oz. = 26 cwt. 14½ lbs. Answer.

(2) Find the inner edge of a cubical cistern that contains 6 cwt. of water, if 1 cubic ft. of water weighs 1000 oz.

Weight of water in cistern = 6 cwt. = 10752 oz.

$$\therefore \text{ volume of water in cistern} = \tfrac{10752}{1000} = 10{\cdot}752 \text{ cub. ft.}$$

Extracting the cube root to two places of decimals we obtain :
Edge of cube = 2·21 ft. = 2 ft. 2½ in. Answer.

(3) The three dimensions of a rectangular parallelepiped are 18 in., 15 in., and 13 in. Find the edge of a cube equal in volume.

Volume of parallelepiped $= 18 \times 15 \times 13$ cub. in. $= 3510$ cub. in.

Extracting the cube root, we obtain edge of cube of equal volume = 15·197 in. Answer.

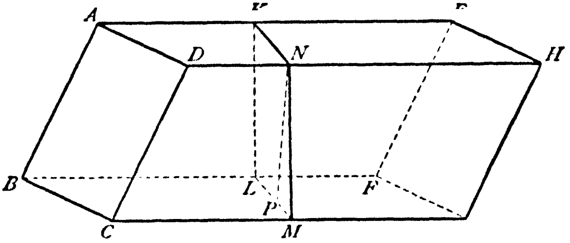
64. *Parallelepiped (oblique-angled).*

Volume $= Bh$ (where B is the base, h the altitude) . . (3).

This result rests on the proposition that *an oblique-angled parallelepiped is equal to a rectangular parallelepiped with the same base and altitude*, a proposition in solid geometry answering to and depending on the proposition in plane geometry, that an oblique-angled parallelogram is equal to a rectangle with the same base and altitude.

It is to be noticed that in the particular case of the rectangular parallelepiped the altitude coincides with the third dimension.

There is another way of expressing the volume of an oblique parallelepiped, which the following figure will illustrate.



Volume of oblique-angled parallelepiped
= *right section* × *length* . . . (4).

The *right section* is the section made by a plane which cuts at right angles any one of the three sets of four parallel edges, any one of the edges so cut being considered the length.

Thus in the figure the right section $KLMN$ cuts the four edges AE, BF, CG, DH at right angles. Thus volume = area of $KLMN$ × length CG.

It is most natural that the right section should cut at right angles the longest dimension, as in the figure; but the result would be the same if the section were drawn at right angles to AB, DC, EF, HG, or to AD, BC, EH, FG.

The result may be easily proved to be identical with the former; for draw the perpendicular NP,

∴ volume = area of $KLMN \times CG$.
= $NP \times LM \times CG$.
= NP × area of $BCGF$.
= altitude × base.

Thus (3) and (4) are identical.

Examples.—(1) In an oblique-angled parallelepiped, given base, right section, and length, find the altitude.

Since volume = altitude × base, and also volume = right section × length,

$$\therefore \text{altitude} = \frac{\text{right section} \times \text{length.}}{\text{base.}}$$

For instance, required to find the vertical depth of a sloping mine-shaft, whose sides are parallel, if the area of the top surface be 300 sq. ft., the cross-section a rectangle, whose sides are 14 ft. and 12 ft., and the length 500 ft.

Here right section = 14 × 12 sq. ft.; base = 300 sq. ft.

$$\therefore \text{vertical depth} = \frac{14 \times 12 \times 500}{300} = 14 \times 20 = 280 \text{ ft.}$$

Answer 280 ft.

(Similarly the length may be determined if cross-section, base and altitude, are given.)

(2) The volume of a parallelepiped is 1980 cubic ft., and the perpendicular distances between the three pairs of parallel planes are respectively 5 ft., 9 ft., 11 ft. Find the total exterior surface.

Let B, B', B'' be the three faces of the parallelepiped, which meet at any angular point. Then, if each of these be taken for base in turn, its perpendicular distance from the opposite face will be the altitude.

Thus we obtain three expressions for the volume:

$$5B \text{ cubic ft., } 9B' \text{ cubic ft., } 11B'' \text{ cubic ft.}$$

$$\therefore B = \tfrac{1980}{5} = 396 \text{ sq. ft.}$$

$$B' = \tfrac{1980}{9} = 220 \text{ sq. ft.}$$

$$B'' = \tfrac{1980}{11} = 180 \text{ sq. ft.}$$

$$\therefore \text{surface} = 2\,(B + B' + B'') = 796 \times 2 \text{ sq. ft.}$$

$$= 1592 = 176 \text{ sq. yds. } 8 \text{ sq. ft.} \quad \text{Answer.}$$

EXAMPLES ON
THE VOLUMES OF PARALLELEPIPEDS

[N.B.—1 gallon of water contains $277\frac{1}{4}$ cubic in.
1 cubic ft. of water weighs 1000 oz.]

(a) Cube

1. Find the volume of a cube whose edge is 3 ft. 5 in.

2. Find the number of cubic ft. in a cube whose diagonal is 21 ft.

3. Find the volume of a cube whose surface is 135 sq. ft. 54 in.

4. The volume of a cube is 4,741,632 cubic yds. Find the edge.

5. The volume of a cube is 10,648 cubic yds. Find the diagonal correct to a yard.

6. The volume of a cube is 10 cubic yds. 15 ft. 559 in. Find the surface.

7. The edge of one cube is 3 times the diagonal of a second; and the sum of their surfaces is 672 sq. in. Find the volumes of each.

8. The sides of three cubes have equal differences, and their sum is 15 in.; the solid content of the three is 495 cubic in. Find their dimensions and volume. (*Sandhurst.*)

9. Find the edge of the cube whose volume is equal to the volumes of three cubes whose edges are 1 ft., 1 ft. 4 in., 1 ft. 8 in.

10. Show that a cube whose volume is 1 cubic ft. is equal to the volumes of three cubes, of which the edge of the first is 6 in., the diagonal of the second $8\sqrt{3}$ in., and the surface of the third $4\frac{1}{8}$ sq. ft.

11. A cubical cistern contains when full 2000 cubic ft. of water. Find the length of one of its sides to the nearest tenth of an inch. (*Sandhurst.*)

12. Find the edge of a cubical vessel which will hold 1 ton's weight of water.

13. Find how many gallons of water a cubical cistern will hold, the area of whose base is 2 sq. ft. 36 in. Answer correct to a gallon.

14. Find the weight of water a cubical vessel will hold whose edge is 2 ft.

15. Two cubical vessels are filled with water. The water in the first weighs 625 oz. less than half the water in the second, and the second contains $39\frac{11}{80}$ more gallons than half the number of gallons in the first. Find the edge of each, assuming that a cubic ft. of water contains $6\frac{1}{8}$ gallons.

16. In a cubical box the thickness of the wood is $\frac{3}{4}$ in., and the edge is 9 in. Find the solid content of the wood.

17. Find the weight of a cubical block of stone, the edge of which is 2 yds., if a cubic yd. weighs 39 cwt. 3 qr. 26 lb. 10 oz.; and find also its cost at £1 3*s.* 4*d.* per ton. Answer correct to the nearest halfpenny.

18. Find the weight of a cubical iron safe whose total exterior surface is 24 sq. ft., and the thickness of the iron is 2 in. (1 cubic in. of iron weighs 4.5 oz.) Show that there are two ways of solving this problem. Which is the most accurate, and why?

19. Find to two places of decimals the number of inches in the edge of a cubical vessel which contains exactly a gallon.

20. A quantity of water contained in a cubical cistern is found to lose by evaporation .04 of its volume in a day. The depth of the cistern is 6 ft., and a cubic ft. of water weighs 1000 oz. Assuming the loss to take place by evaporation only, find to one decimal place what weight of water will be left in the cistern after the expiration of 10 days. Given log 2 = .3010300, log 3 = .4771213, log 14360 = 4.1571544, log 14361 = 4.1571847. (*Sandhurst.*)

(b) Rectangular Parallelepiped

21. The dimensions of a rectangular parallelepiped are 2 ft. 1 in., 8 yds. 2 ft., and 128 ft. Find the volume.

22. The base of a rectangular parallelepiped contains 73 sq. ft. 18 in., and the height is 12 ft. 6 in. Find the volume.

23. The length and breadth of a rectangular parallelepiped are 12 ft. and 3 ft., and the diagonal is 13 ft. Find the volume.

24. The base of a rectangular parallelepiped is 1 sq. ft. 141 sq. in., and the volume is 5 cubic ft. 1050 in. Find the altitude.

25. Given the volume = 73 cubic yds. 14 ft. 504 in., breadth = 8 ft. 10 in., and height = 7 ft. 9 in., find the length.

26. Given the volume = 5 cubic yds. 2 ft. 1344 in., and the height = 10 ft. 8 in., find the base.

27. Given the three dimensions, 108 yds., 294 yds., and 504 yds., find the edge of a cube equal in volume.

28. The three different edges of a rectangular parallelepiped are 3, 2.52, and 1.523 ft. in length. Find the number of cubic ft. in the figure. Find also the cubic space inside a box of the same extreme dimensions, constructed of a material of $\frac{1}{10}$ of a foot in thickness. (*Sandhurst.*)

29. Two of the faces of a rectangular parallelepiped contain 30 sq. in. each, two more 24 sq. in. each, and two more 20 sq. in. each. Find the volume.

30. Find the weight of a solid bar of gold 1 ft. long, whose cross-section is a square (side = 2 in.), if a cubic ft. of gold weighs 19,300 oz. (avoirdupois).

31. Find the volume of air in a rectangular room, whose dimensions are 17 ft. 4 in., 11 ft. 6 in., and 7 ft. 7 in.

32. A rectangular room contains 1125 cubic ft. of air, and its length, breadth, and height are proportional to 6, 4, 3. Find the height.

33. A rectangular room contains 2625 cubic ft. of air. The expense of carpeting the floor, at 6*s.* 3*d.* per sq. yd., is £10 18*s.* 9*d.* Find the height.

34. What value of bricks, 9 in. × $4\frac{1}{2}$ in. × 3 in., will be required to build a wall 125 yds. long, 7 ft. high, and 1 ft. $1\frac{1}{2}$ in. thick., if bricks cost a guinea per thousand?

35. A wall, 8 ft. 6 in. high, and 1 ft. 3 in. thick, surrounds a square garden, whose area is $\frac{9}{10}$ acre. Find the number of bricks, 9 × 5 × 3 in., which are required to build the wall, allowing for two doors 3 ft. 9 in. wide and 6 ft. 9 in. high.

36. Find the number of cubic yds. of earth which must be dug out to make a ditch 3 ft. deep, $4\frac{1}{2}$ ft. wide, and 108 yds. long.

37. If a man can dig out 10 cubic yds. of earth per hour, how long will he take in digging out a ditch 120 yds. long, the cross-section being a rectangle containing $10\frac{1}{2}$ sq. ft.?

38. The cross-section of a square piece of timber is 150 sq. in. How many feet must be sawn off the length so that the part sawn off may contain 3 cubic ft. 216 in.?

39. A cubic ft. of gold is beaten out into gold leaf $\frac{1}{250,000}$ in. thick. Find the area which it will cover.

40. If a plate of gold, whose dimensions are 18 in., 10 in., and 9 in., be beaten out into gold leaf to cover 500,000 sq. yds., find the thickness of the leaf.

41. Find the cost of excavating to the depth of 13 ft. the foundations of a building, which stands on a rectangular area of 2500 sq. ft., at 1*s.* $1\frac{1}{2}$*d.* per cubic yard.

42. A bar of metal, 9 in. wide, 2 in. thick, and 8 ft. long, weighs 1 lb. to the cubic inch. Find the length and thickness of another bar of the same metal, width, and solid content, if 2 in. cut off from the end weigh 27 lb. (*Sandhurst.*)

43. A rectangular safe is to be made of iron $1\frac{1}{2}$ in. thick, so as to contain 16 cubic ft. inside. If the interior height = breadth = $\frac{1}{2}$ length, find its total exterior surface.

44. A rectangular box contains 16 barrels' weight of gunpowder. The inside of the box is to be lined with copper 13½ oz. to the sq. ft., and 2*s*. a lb. If the height = breadth = ½ length (internal dimensions), find the cost of the lining. A barrel of gunpowder contains 128 lb., each of which occupies 32 cubic in.

45. The external length, breadth, and height of an open rectangular metal cistern are 6 ft. 6 in., 5 ft. 4 in., and 2 ft. 9 in., and the thickness of the metal is ½ in. When empty the weight is 3953⅛ lb.; when filled with sand it weighs 15,182 lb. 4⅔ oz. How much heavier is the metal composing it than its own bulk in sand?

46. A rectangular tank has a square base, the side of which is 7½ ft. What weight of water must be drawn off so as to make the surface sink 4 in.?

47. A rectangular tank is 17 ft. 3 in. broad and 23 ft. 1¼ in. long. If 621 gallons of water are drawn off, how much will the water sink?

48. How much water will a rectangular vessel hold, whose dimensions are length 10 ft. 6 in., breadth 8 ft. 3 in., height 3 ft. 3 in.? Find answer correct to nearest gallon.

49. A rectangular tank has a square base, the side of which is 16 ft. Find correct to the nearest gallon how much water must be poured into it so as just to cover a cubical block of wood lying in it, measuring 16 in. each way.

50. A rectangular tank is 15 ft. long and 11 ft. 4 in. broad. If 59½ cubic ft. of sand be thrown in, how much will the water rise?

51. The area of the base of a rectangular tank is 120 sq. ft. Find the solid content of a body which, when thrown into the water, makes the surface rise 1½ in.

52. If a body 30 cubic ft. in volume, thrown into the water, makes the surface rise 4 in. in a rectangular tank, what is the area of the base?

53. Water is poured at the rate of 360 gallons a minute into a rectangular reservoir, the base of which is 12,000 sq. ft. Calculate the time in which the surface of the water will be raised 6 in. Answer correct to nearest minute.

54. A rectangular reservoir, whose dimensions are 125 ft. and 77 ft., and depth 9 ft., is being emptied by two pumps, one of which pumps out 1000 gallons in three minutes, and the other 1000 gallons in $2\frac{1}{3}$ minutes. How long will they take to empty the reservoir? And if the former stops when the reservoir is $\frac{3}{4}$ empty, how long will the other take to empty the remainder?

55. The water supply of a town is 148,000 gallons a day, and is taken from 3 rectangular reservoirs, which are always kept on an equal level. If the bases of the three contain 15,000, 17,000, and 18,000 sq. ft. respectively, how much would the water removed lower the surface every day?

56. At what rate of speed must the water flow through a pipe (the section of which is a square, side = $2\frac{1}{2}$ in.) into a reservoir, from which 96,000 gallons of water are taken every day, in order to keep the water permanently on the same level?

57. How many gallons of water are drawn off daily from a rectangular reservoir, the area of whose base is 11090 sq. yds., if the surface of the reservoir sinks 2 in. a day?

58. Six iron tubes, the cross-section of any one of them being a square, are joined together so as to form a hexagonal ring, which can be filled with water. If the area of a cross-section = 16 sq. in., and distance between any two internal opposite corners of the hexagon = 1 yd., find whole surface of tubing, and its content, neglecting thickness of iron. (*Sandhurst.*)

59. If the sum of all the edges of a rectangular parallelepiped = $4p$, the surface = $2q$, and the volume = r, show that the three dimensions are the roots of the equation

$$x^3 - px^2 + qx - r = 0.$$

(c) Oblique-angled Parallelepiped

60. If the base of a parallelepiped is 49½ sq. ft., and the altitude is 19 yds. 9 in., find the volume.

61. If the volume is 99 cubic ft. 1296 in., and the base is 7 sq. ft. 56 in., find the altitude.

62. The base of a parallelepiped is 252 sq. in., and the altitude is 1½ ft. If a side-face 378 sq. in. in area be taken for base, what would be the altitude?

63. The base of a parallelepiped is 96 sq. ft., and the perpendicular distances between the three pairs of parallel planes are respectively 8 ft., 12 ft., and 16 ft., the smaller value being the altitude. Find the total surface.

64. The volume of a parallelepiped is 5040 cubic ft., and the perpendicular distances between the three pairs of parallel planes are respectively 14 ft., 18 ft., and 20 ft. Find the total surface.

65. The length of a parallelepiped is 17 ft. 6 in., and the right section is 1 sq. ft. 6 in. Find the volume.

66. The volume of a parallelepiped is 124 cubic ft. 54 in., and the right section is 7 sq. ft. 126 in. Find the length.

67. The right section of a parallelepiped is 2 sq. ft. 41 in., the length 8 in., and the altitude 7 in. Find the base.

68. Find the length of a pipe which will hold 160 gallons of water, if the cross-section is a rectangle 4 in. by 2½ in.

69. The horizontal section of a sloping pipe is a parallelogram 22.18 sq. in. in area. Water is pumped from below to a higher level of 40 ft. Find the number of gallons in the pipe.

70. The horizontal section of a sloping mine-shaft is 317 sq. ft. in area. If the lower end of the shaft is 200 yds. vertically below the upper end, find the number of cubic ft. of air in the shaft. Also if the cross-section is a rectangle 25 sq. yds. in area, find the length of the shaft.

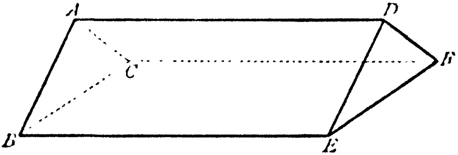
CHAPTER III.—THE PRISM

Section I.

65. DEFINITION 10.—"*A prism is a solid figure contained by plane figures, of which two that are opposite are equal, similar, and parallel to each other; and the others are parallelograms.*" (Euclid, xi. def. 13.)

The two equal and opposite figures are called the *bases* of the prism; the parallelograms are called the *side-faces*, and their intersections the *edges*.

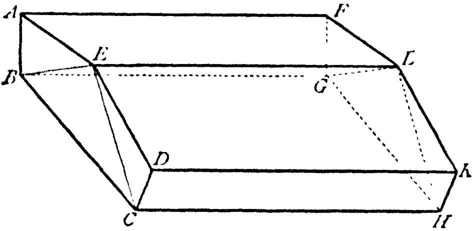
For instance, let *ABC*, *DEF* be two equal and similar triangles, situated so that their sides are respectively parallel each to each. Join *AD*, *BE*, *CF*. Then the solid figure



ABCDEF is a prism, contained by five faces, i.e. the two triangles *ABC*, *DEF*, and the three parallelograms *ABED*, *CBEF*, *ACFD*. The triangles are called its bases; the parallelograms its side-faces, and their intersections *AD*, *BE*, *CF* are its edges.

N.B.—It is obvious (Euclid, i. 33) that *AD*, *BE*, *CF* are all equal. Hence it is customary to call either of them the *edge* of the prism. They are also called the *length* of the prism.

66. When the base of a prism is a triangle, the prism is called *triangular*. But the base of a prism can be *any plane figure whatever*. The subjoined figure gives an illustration of a *pentagonal prism*, with a pentagon for its base. The side-faces and edges of the prism will obviously be each the same in number as the sides of the base. Thus in the illustration there are five sides and five edges.



Any prism on a polygonal base can be divided into triangular prisms. For instance, by drawing planes through *EBGL* and *CELH*, we divide the above prism into three triangular prisms.

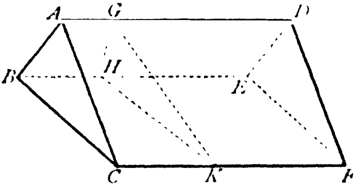
67. A prism is called *right* when the edges are at right angles to the planes of the bases If the edges are not at right angles to the bases, the prism is called *oblique*. We shall be principally concerned with *right prisms*. In a right prism the sides are *rectangles*.

68. If the bases of a prism are parallelograms, then, as the sides also are parallelograms, the figure is contained by six parallelograms. These are readily seen to lie opposite to each other in pairs, each pair being equal and parallel. Therefore the figure is a *parallelepiped*, which we thus perceive to be a particular case of the prism.

N.B.—A caution is needed about *oblique-angled* parallelepipeds. An oblique-angled parallelepiped may yet be a *right* prism, if the side-faces are rectangles, and only the bases oblique-angled parallelograms. Thus the terms 'oblique-angled' and 'oblique' must not be identified, though usually they coincide.

69. DEFINITION 11.—*A frustum of a prism is that part of it which is cut off by a plane not parallel to its base.*

For instance, let the triangular prism *ABCDEF* be cut by the plane *GHK*, not parallel to *ABC*. Then the prism is divided into the two frusta *ABCGHK* and *GHKDEF*. The plane *GHK* must not be parallel to *ABC*; for if it were, the triangle *GHK* would be similar and equal to the triangle *ABC*, so that *ABCGHK* would simply be another prism. Hence a plane parallel to the base of a prism divides it into two prisms with the same base.



70. The *altitude* of a prism is the perpendicular dropped from any point in one base on to the other.

In the case of a right prism the altitude obviously coincides with the length.

71. The exterior surface of any prism is the sum of the plane figures containing it, whose area can be found by the rules given for plane surfaces. Thus:

(1) Exterior surface of *prism* = two plane figures + a certain number of parallelograms.

(2) Exterior surface of *frustum of prism* = two plane figures + a certain number of trapezoids.

Examples.—(1) Find the total exterior surface of a right triangular prism, the sides of the base being 2 ft. 1 in., 2 ft. 5 in., and 3 ft., and the altitude 2 ft.

Area of base $= \sqrt{45 \cdot 20 \cdot 16 \cdot 9} = 30 \cdot 4 \cdot 3 = 360$ sq. in.
$\therefore$ area of two bases $= 720$ sq. in. $= 5$ sq. ft.
Area of side-planes $=$ perimeter of base × altitude.
$= 90 \times 24$ sq. in. $= 15$ sq. ft.
$\therefore$ total surface $= 20$ sq. ft. Answer.

(2) Find the cost of painting, at 4*d.* per sq. ft., the walls of a regular hexagonal room whose side = 9 ft. 2 in., and height 10 ft. 3 in.

The problem is to find the area of the side-planes of a prism on an hexagonal base. Using the result of the last example, we have:

Area of walls $=$ perimeter × height.
$= 6 \times 9\frac{1}{6} \times 10\frac{1}{4}$ sq. ft.
$= \frac{55 \times 41}{4}$ sq. ft.
$\therefore$ cost of painting $= 55 \times 41$*d.* $= 2255$*d.*
$=$ £9 7*s.* 11*d.* Answer.

(Compare with this the examples on the rectangle, bk. i. chap. ii. We now see that the formula, area of walls = perimeter × height, applies in the case of every room where the walls are plane surfaces. It is clear that in such a case the walls may always be considered as placed side by side in the form of a rectangle, whose length is the perimeter.)

EXAMPLES ON THE PRISM

(LENGTHS AND SURFACES)

1. Find the total exterior surface of a right prism whose base is a regular pentagon (side = 8 ft.), and whose altitude is 5 ft.

2. Find the area of the whole surface of a right triangular prism, the sides of whose base are 1 ft. 1 in., 1 ft. 2 in., and 1 ft. 3 in., and whose altitude is 1 ft.

3. Find the exterior surface of an oblique prism whose base is a regular hexagon (side = 2 ft.), and whose total length is 12 ft, given that the plane angles which form one of the solid angles are each 60°.

4. The base of a right prism is a trapezium whose parallel sides are 16 in. and 11 in., and third side is 1 ft. If the length of prism is 2 ft. 3 in., find the total exterior surface.

5. A right prism stands on a triangular base, and every edge is 2 ft. Find its total exterior surface.

6. Find the total exterior surface of a right prism 1 ft. 6 in. in length, whose base is an isosceles triangle, of which the base is 2 ft., and the perpendicular on it from vertex 9 in.

7. Find the perimeter of the base of a right prism whose length is 12 ft. 9 in., and the total area of whose side-planes is 127 sq. ft. 72 in.

8. The base of a right prism is triangular, and the side-planes are all squares. If the total surface is 61 sq ft., find the area of the base.

9. A right triangular prism has all its edges equal, and its exterior surface is 30½ sq. ft. Find its edge.

10. The surface of a right prism on a regular hexagonal base is equal to that of a cube whose edge is 8 ft. If the side of hexagon is 4 ft., find altitude of prism.

11. A right prism has an equilateral triangle for base whose side is 5 in. A frustum is cut off by a plane, which leaves the parallel edges 6 in., 9 in., and 10 in. Find the total exterior surface of frustum correct to $\frac{1}{10}$ of a sq. in.

12. The base of a right prism is an equilateral triangle. The area of the side-planes is 27 sq. yds. 2 ft., and height is 8 ft. 2 in. Find the area of the base correct to $\frac{1}{10}$ of a sq. ft.

13. Find the edge of the cube whose surface is one-third of that of the right prism whose base is a right-angled triangle (sides 12 ft. and 16 ft.), and whose height is 25 ft.

14. Find the lateral surface of eight octagonal stone pillars, the height of each being 22 ft. 7 in., and the side of each octagon 1 ft. 3 in.

15. Find the cost of papering the walls of a regular octagonal chamber with paper 30 in. wide, at 2½*d.* per yd., if the height is 10 ft., and side of the octagon 7½ ft.

16. Find the cost of painting the walls of a hexagonal room, if the height is 9 ft. 4 in., and each side of the hexagon is 10 ft. 1 in., the painting to cost 7½*d.* per sq. ft.

17. Find how many sq. ft. of tin are required to construct a pipe with a hexagonal bore, each side of hexagon being 1½ in., and the length 40 ft.

Sec. II.—Volumes

[FORMULÆ:

(1) Volume of *any prism* = base × altitude.

(2) Also volume of *oblique prism* = right section × length.

(3) Also volume of *triangular prism*
$= \frac{1}{2}$ any side-face × perpendicular from opposite edge.

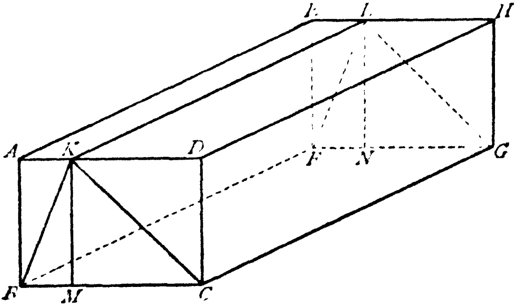
(4) Volume of *frustum of right triangular prism*
= base × $\frac{1}{3}$ sum of parallel edges.

Volume of *frustum of oblique triangular prism*
= right section × $\frac{1}{3}$ sum of parallel edges.]

72. *Volume of Prism* = Bh.

This formula is true both for right prisms and for oblique prisms. In the case of the right prism the altitude is identical with the length. This case of the formula is susceptible of an easy proof, based on the result obtained for the rectangular parallelepiped.

Let $ABCDEFGH$ be a rectangular parallepiped. In AD take any point K, and draw KL parallel to the edges of the figure. Join KB, KC, LF, LG. Then $KBCLFG$ is a right triangular prism, and its volume = $\frac{1}{2}$ the parallelepiped.



For draw the perpendiculars KM, LN, and let a plane pass through them, then the plane passing through $KBLF$ obviously divides in half the parallelepiped $ABMKEFNL$; and the plane passing through $KCGL$ obviously divides in half the parallelepiped $KMCDLNGH$.

$\therefore$ the prism $KBCLFG$ is half of the whole parallelepiped.

$\therefore$ volume of prism $= \frac{1}{2}$ base $ABCD \times$ altitude.

But triangle KBC (which is the base of the prism) $= \frac{1}{2}$ base $ABCD$.

$\therefore$ volume of prism = base × altitude.

Thus the formula holds for a right triangular prism. It follows that it holds for *any* right prism; for a prism on any base can always be divided in a number of prisms on triangular bases, all having the same altitude.

$\therefore$ volume of right prism = sum of these bases × altitude.

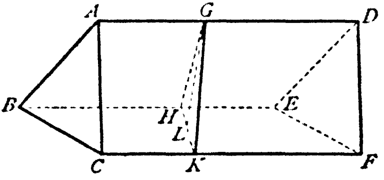
But the sum of these triangular bases = base of the prism.

$\therefore$ volume of right prism = base × altitude . . (1).

73. The volume of an oblique-prism can be shown equal to the volume of a right prism with the same base and altitude. Hence the formula also holds for an oblique prism.

But the *volume of an oblique prism also = right section × length.* (2.)

A right section is a section made by a plane cutting all the parallel edges at right angles. Thus in the figure



Volume of prism = triangle $GHK \times CF$.

For if we suppose the plane through GHK to cut the prism into two parts, and that the left-hand part be taken up, and fitted on to the other side, so that ABC falls exactly on DEF, then we have a right prism of the same volume and length, whose base is the triangle GHK.

74. *The volume of a triangular prism* also = ½ *side-face* × *perpendicular from opposite edge.* (3.)

This formula may be easily proved in the case of a right triangular prism. Referring back to the figure in paragraph 1, we have:

Volume of parallelepiped = base $BCGF$ × height KM.

∴ volume of prism = ½ $BCGF \times KM$.

But $BCGF$ is a side-face of the prism, and KM is the perpendicular on it from the opposite edge KL.

In the case of an oblique triangular prism the formula may be deduced from the one in the last paragraph. For in the last figure

Volume of prism triangle $GHK \times CF$.

$= \frac{1}{2} GL \times HK \times CF$.

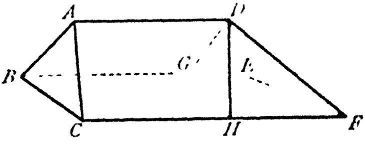
$= \frac{1}{2} HK \times CF \times GL - \frac{1}{2} BCFE \times GL$.

= ½ side-face × perpendicular from opposite edge.

75. *Frustum of right triangular prism.*

Volume of frustum base × ½ *sum of parallel edges.* (4)

Let $ABCDEF$ be a frustum of a right triangular prism, cut by a plane DEF not parallel to the base ABC. Then the parallel edges are those cut off by the plane DEF (namely, AD, BE, and CF), and volume of frustum



= base $ABC \times \frac{1}{2} (AD + BC + CF)$.

The parallel edges are sometimes denoted by h', h'', h'''. Thus the volume may be written $= \frac{1}{2} (h' + h'' + h''') \times$ base.

By drawing a plane through D parallel to ABC, we divide the frustum into a right prism and a pyramid. The sum of these will give the volume of the frustum. (See next chapter.)

A little consideration will show that this formula cannot be extended to a prism on any base. The frustum of a prism on a polygonal base can be divided into triangular frusta, which may be summed severally.

The volume of the frustum of *an oblique triangular prism* = *area of right section* × $\frac{1}{3}$ *sum of parallel edges.* This may be expressed as before:

$$\text{Volume} = \tfrac{1}{3}\,(h' + h'' + h''') \times \text{right section}.$$

76. In the frustum of an oblique triangular prism, if one of the side-planes is a rectangle, the frustum is called a *wedge*, of which the rectangle is the *base*, and the side opposite to it the *edge*.

If a and b are the length and breadth of the rectangle, c the edge, and h the altitude,

$$\text{Volume of wedge} = \frac{bh}{6}\{2a + c\}.$$

Also, if we cut a wedge by a plane parallel to its base, we get a figure resembling a prism, contained by two rectangles and four trapezoids. This figure is called a *prismoid.*

If a, b, a', b' be length and breadth of the two rectangles, and h the altitude,

$$\text{Volume of prismoid} = \frac{h}{6}\{ab + a'b' + (a + a')\,(b + b')\}.$$

Examples.—(1) Find, correct to a cubic ft., the number of feet of air in a regular octagonal tower whose inside perimeter is 35 ft., and whose height is 60 ft.

Area of base $= 2\,(\sqrt{2} + 1)\,.\,(\frac{35}{8})^2$ sq. ft. (See bk. i. chap. iv.)

Taking $\sqrt{2} = 1{\cdot}4142$, and working out, we have:

Area of base $= 92{\cdot}42$ sq. ft.

$\therefore$ volume of prism $= 92{\cdot}42 \times 60$ cubic ft.

$= 5545{\cdot}2$ cubic ft.

$\therefore$ volume of air correct to a cubic ft. $= 5545$ cubic ft.

(2) Find the volume of an oblique triangular prism whose length is 6 ft, if the sides of the right section are 2 ft. 5 in., 2 ft. 11 in., and 4 ft. respectively.

Here right section is a triangle whose area

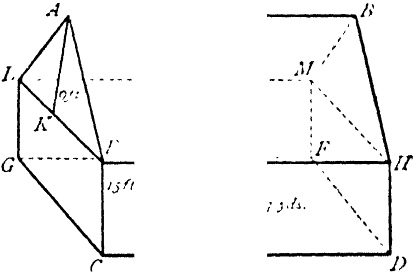
$$= \sqrt{\underset{7}{\cancel{56}} \,.\, 27 \,.\, \underset{3}{\cancel{21}} \,.\, \cancel{8}} = 8 \times 7 \times 9 \text{ sq. in.} = 504 \text{ sq. in.} = 3\tfrac{1}{2} \text{ sq. ft.}$$

$\therefore$ volume of prism $= 3\frac{1}{2} \times 6 = 21$ cubic ft. Answer.

(3) Find the number of cubic ft. of air in a building with an open roof, the height of the side-walls being 15 ft., and of the gable 24 ft., and the area of the floor being 100 sq. yds.

Volume of house = prism + parallelepiped.

In parallelepiped, base $GCDF = 100$ sq. yds.



$\therefore$ altitude $CE = 15$ ft.

$\therefore$ volume of parallelepiped $= 900 \times 5$ cubic yds.

Again, in prism, volume of side-plane $LEHM = 100$ sq. yds., and perpendicular $AK = 24 - 15 = 9$ ft.

$\therefore$ volume $= \frac{1}{2} \,.\, 900 \times 3$ cubic yds. Formula (3).

$\therefore$ whole volume $= 100\,(5 + 1\frac{1}{2}) = 100 \times 6\frac{1}{2}$.

$= 650$ cubic yds. Answer.

(4) Find the volume of the frustum of a prism, whose base is a right-angled triangle whose sides are 1 ft. 3 in. and 2 ft. 1 in. if the parallel edges are 5, 7, and 10 in. respectively.

Area of base $= \frac{1}{2} \times 15 \times 25$ sq. in.

and sum of parallel edges $= 22$ in.

$$\therefore \text{volume} = \frac{1}{3} \times \overset{11}{\cancel{22}} \times \frac{1}{\cancel{2}} \times \overset{5}{\cancel{15}} \times 25 = 125 \times 11 \text{ cubic in.}$$

$= 7$ sq. ft. 79 cubic in. Answer.

EXAMPLES ON THE VOLUMES OF PRISMS

1. Find the volume of a prism whose base is 134 sq. yds. 8 ft. 36 in., and whose altitude is 21 yds. 1 ft. 4 in.

2. Find the volume of a triangular prism, the sides of whose base are 6 ft. 6 in., 7 ft., and 7 ft. 6 in., and whose altitude is 5 yds.

3. Find the volume of a triangular prism whose altitude is 10 yds. 2 ft., if one side of the base is 2 ft. 9 in., and the perpendicular on it from the opposite angle is 2 ft. 6 in.

4. Find the volume of a triangular prism, if one of the side-planes is 42 sq. yds. 7 ft. 108 in. in area, and the perpendicular on it from the opposite edge is 3 yds. 1 ft. 8 in.

5. A prism stands on a regular hexagon for base, and all its side-faces are squares. Find the surface and volume, if the side of each square is 1 ft.

6. Find the volume of an oblique prism whose right section is 26 sq. yds. 3 ft. 19 in., and length is 4 yds. 1 ft. 5 in.

7. Find the volume of an oblique prism whose length is 17 ft. 6 in., and whose right section is a trapezoid whose parallel sides are 12 ft. 5 in. and 7 ft. 7 in., and whose altitude 6 ft. 8 in.

8. The volume of a prism is 550.296 cubic in., and its base is 76.43 sq. in. Find the altitude.

9. A right triangular prism has all its edges equal, its volume is 30 cubic ft. Find its edge in feet and decimal of a foot.

10. The right section of an oblique prism is 11 sq. yds. 1 ft., and the altitude is 12 ft. 6 in. If the base be 12 sq. yds., find the length.

11. The sides of the base of a right triangular prism are 7 ft. 7 in., 8 ft. 2 in., and 8 ft. 9 in. Find the volume of a frustum, the sum of the parallel edges of which is 18 yds.

12. The area of the base of a right triangular prism is 72 sq. in. Find the volume of a frustum, if the parallel edges are 3⅚ in., 5⅙ in., and 6⅔ in.

13. The area of the base of a triangular prism is 42 sq. yds. 7 ft. Find the volume of a frustum cut by a plane which cuts the edges at altitudes of 10 ft. 9 in., 12 ft. 11 in., and 16 ft. 4 in. respectively above the plane of the base.

14. The parallel edges in the frustum of an oblique prism are 2 ft. 7 in., 3 ft. 1 in., and 3 ft. 4 in., and the right section is a triangle whose sides are 3 ft. 5 in., 4 ft. 3 in., and 4 ft. 10 in. Find the volume of the frustum.

15. Find the volume of the frustum of a right prism, the base of which is a parallelogram whose sides are 3 ft. and 2 ft., and whose included angle is 30°, if two opposite parallel edges are 3 ft. and 4 ft.

16. Show how to find the volume of a frustum of a right prism on a pentagonal base.

17. The base of a solid prism is 8 sq. yds. 8 ft., and the altitude is 16 ft. Find the weight, if 1 cubic ft. weighs 350 lb.

18. Find the weight of a block of stone, in shape a frustum of a right triangular prism, the base of which is 23 sq. ft. 21 in., and the sum of the parallel edges 14 ft. 9 in., if 3 cubic feet of stone weigh 4 cwt.

19. Find the height of a regular octagonal tower, which contains 7040 cubic ft., and whose internal perimeter is 36 ft.

20. Find to the nearest lb. the weight of an octagonal stone pillar, whose height is 15 ft., if the side of the octagon is 1 ft., and a cubic ft. of stone weighs 166 lb.

21. How many gallons of water will a ditch hold which is 54 ft. long, 6 ft. deep, 3 ft. 4 in. broad at the bottom, and 5 ft. 2 in. broad at the top?

22. How deep is the water in a ditch which holds 6400 gallons, if the length be 50 ft., the breadth at the bottom 3 ft. 6 in., and the breadth at the top 5 ft.? Answer correct to the nearest inch.

23. The cross-section of a water-pipe is a regular hexagon, whose side is 3 in. At what rate must water flow through the pipe in order to fill in 15 hours a rectangular reservoir, whose base is 10,000 sq. ft., and whose depth is 9 ft.?

24. The longer sides of a bath slope equally to the bottom, which is level, the other two sides are perpendicular. Given length of bath = 134 ft., breadth at top = 42 ft., breadth at bottom = 30 ft., and depth = 8 ft. 6 in., find number of gallons of water the bath will hold.

25. A vessel, in shape a prism, on a regular hexagonal base, whose side is 4 in., is filled with water. Find how much the water will sink if ½ pint is taken away. Answer to three decimal places of an inch.

26. An embankment 13½ ft. high is 20 ft. broader at the bottom than at the top. Find what its breadth at the bottom must be in order that a quarter of a mile's length may contain 21,120 cubic yds. of earth.

27. Find the number of cubic ft. of air in a rectangular building with an open roof, the height of the side walls being 12 ft., and of the gables 19 ft., and the area of the floor being 76 sq. yds. (*Sandhurst.*)

28. How high must the top of an open roof rise above the sides of a rectangular room, whose floor contains 83⅓ sq. yds., in order to add 3006 cubic ft. to the solid content of the room?

29. The vertical ends of a hollow trough are parallel equilateral triangles, with 12 in. in each side, the bases of the triangles being horizontal. If the distances between the triangular ends be 6 ft., find (1) the number of cubic ft. of water the trough will contain, (2) the number of gallons it will contain, it being given that a gallon of water weighs 10 lb., and a cubic ft. of water 62.5 lbs. (*Sandhurst.*)

30. The cross-section of a hall is a rectangle surmounted by a triangle. If the breadth and total height are each 20 ft., and the height of the side walls 14 ft., and the hall contains 20,400 cubic ft. of air, find the cost of decorating the wall and ceiling at 3*s.* 6*d.* per sq. ft.

CHAPTER IV.—THE PYRAMID

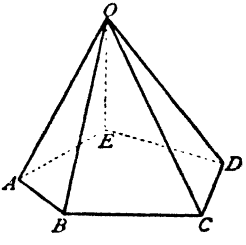
Section I.

77. DEFINITION 12.—"A pyramid is a solid figure contained by planes which are constructed between one plane, and one point above it at which they meet." (Euclid xi. def. 12.)

The definition will be made clearer if we substitute *plane figure* for plane. It may then be stated as follows:

A pyramid is a solid figure contained by plane rectilinear figures, one of which can have any number of sides, while the others are triangles which have a common vertex in some point above the plane of the first-mentioned figure.

The first-mentioned figure is called the *base*, and the common vertex of the triangles the *vertex* of the pyramid; the triangles are the *side-faces*, and their intersections the *edges*.



For example, let *ABCDE* be a pentagon. Take any point *O* above the plane of *ABCDE* and join *O* to the angular points of the pentagon. Then the solid figure *OABCDE* will be a pyramid. *O* is the vertex; *ABCDE* the base; the triangles *OAB*, &c., the side-faces; and *OA*, *OB*, &c., the edges.

78. The following facts about the pyramid should be carefully noticed:

(1) The *base* may be *any* rectilinear figure whatever, regular or irregular. In the figure above we took an irregular pentagon for the base, but we might just as well have taken a triangle, a rhombus, or a regular hexagon. A pyramid is always named after the shape of its base—triangular, square, pentagonal, &c. It will be observed the number of side-faces and edges of the pyramid will always each be equal to the number of sides in the base.

(2) The *vertex* may be *any* point whatever above the plane of the base. We have followed Euclid in retaining the distinction given in the last six words, but it is really of no importance. If the point be taken *below* the plane of the base we can talk of an *inverted pyramid.* But the student will see at once that a pyramid may be taken in any position. For instance, in the triangular pyramid drawn in the next section but one we might with equal propriety call A the vertex and OBC the base.

But it is really important to observe that the vertex may be any point. For instance, it may be so taken as to make one of the side-faces perpendicular to the base, or so as to make all the edges cut the base at a very acute angle. It is obvious, therefore, that *the edges and the side-faces need not be all equal.*

79. The student's attention is directed to these points to impress on his mind that the pyramid alone, of all the solid figures we have to consider, has no natural regularity about it. We may consider how far regular a pyramid can be.

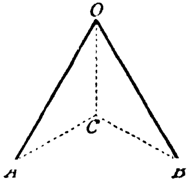
First. The base may be a regular polygon. The pyramids of Egypt, whose bases are squares, are illustrations of this.

Secondly. The vertex may be so taken that all the edges are equal.

A pyramid which complies with both these conditions may be called *symmetrical.* In a symmetrical pyramid all the side-faces are equal, and equally inclined to the base.

80. *Tetrahedron.*[*]

In one case the pyramid is wholly regular, all the faces being similar and equal. This is in the tetrahedron.



DEFINITION 13.—"A tetrahedron is a solid figure contained by four equal and equilateral triangles." (Euc. xi. def. 26.)

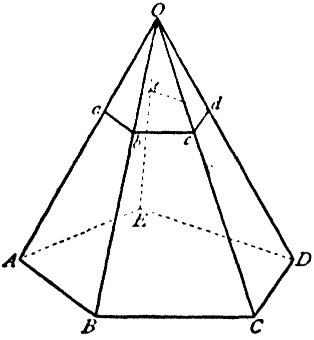
Any one of the triangles can be taken for base, and the opposite angular point will be the vertex.

It is obvious in the tetrahedron that all the edges are equal, and consequently that the figure is wholly determined when the edge is known.

81. *Frustum of a pyramid.*

DEFINITION 14.—A frustum of a pyramid is that part of it which is cut off between the base and a plane parallel to it.

For instance, let the pentagonal pyramid *OABCDE* be cut by the plane *abcde* parallel to the base *ABCDE*. Then the pyramid is divided into two parts, of which *Oabcde* is a pyramid, and the lower part the frustum of a pyramid.



If the plane *abcde* were not parallel to *ABCDE* we should obtain a more complicated figure, to which the name frustum of a pyramid might be given; but it is better to restrict the name as above.

Since the two parallel planes *abcde*, *ABCDE* are cut by the plane OAB, ab and AB are parallel. Therefore the triangles Oab, OAB are similar, and

---

[*] The word *tetrahedron* I use in Euclid's sense, not recognizing the necessity of extending its meaning so as to make it synonymous with triangular pyramid.

$$ab : Ob :: AB : OB.$$

$$\text{Similarly } Ob : bc :: OB : BC,$$

$$\text{and ex æquali } ab : bc :: AB : BC.$$

Thus the sides of $abcde$ are proportional to those of $ABCDE$, and, as they are also parallel, the polygons $abcde$, $ABCDE$ are similar.

Hence the top surface of the frustum of a pyramid is similar to the base.

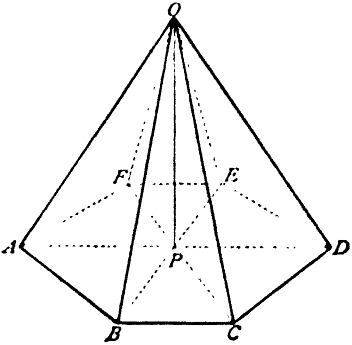
(The top surface and base are sometimes called the bases of the frustum.)

82. The *altitude* of a pyramid is the perpendicular from the vertex on the base, or on the plane of the base produced.

The altitude of the frustum of a pyramid is the perpendicular dropped from any point in the top surface on to the base.

Altitude of a symmetrical pyramid.

Let $OABCDEF$ be a symmetrical pyramid, having a regular polygon for its base, and all its edges, OA, OB, &c., equal. Draw OP perpendicular to its base, and join P to the angular points of the polygon. Then the angles OP makes with these straight lines are right angles.



$\therefore AP^2 = OA^2 - OP^2 = OB^2 - OP^2 = BP^2.$

$\therefore AP = BP$, and similarly it can be proved that all the six straight lines PA, PB, PC, PD, PE, PF are equal.

$\therefore P$ is the centre of the circle circumscribing the polygon.

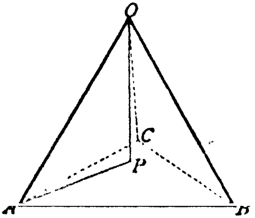
Hence the altitude of a symmetrical pyramid is the straight line joining the vertex to the centre of the circle circumscribing the base.

83. The *exterior surface of a pyramid* = sum of the figure which is the base, and a certain number of triangles.

The exterior surface of the frustum of a pyramid = sum of the two similar figures which are its bases, and a certain number of trapezoids.

In all cases examples can be solved by the rules given for plane surfaces.

Examples.—(1) Find the altitude and surface of a tetrahedron whose edge is 1 ft.



Let $OABC$ be a tetrahedron. Draw OP the altitude; then P is the centre of the circumscribed circle to ABC.

$$\therefore AP = \frac{AB}{2 \sin 60^\circ} = \frac{1}{\sqrt{3}} \text{ ft.}$$

$$OP = \sqrt{OA^2 - AP^2} = \sqrt{1 - \tfrac{1}{3}}$$

$$= \sqrt{\tfrac{2}{3}} \text{ ft.}$$

$= 9{\cdot}8$ in. (very nearly). **Answer.**

Surface $= 4$ times $ABC = \sqrt{3}$ sq. ft. $= 1{\cdot}732$ sq. ft. **Answer.**

(2) The altitude of a symmetrical hexagonal pyramid is 11 ft., and the side of the base is 8 ft. Find the whole surface.

In the figure on p. 149, draw OG at right angles to AB, and join GP. Then PG will be at right angles to AB, and there-

fore (bk. i. chap. iv.) will be the *radius of the circle inscribed in the hexagonal base.*

$$\therefore\ PG = \frac{AB}{2 \tan 30^\circ} = \frac{\sqrt{3}}{2} \,.\, 8 \text{ ft.} = 4\sqrt{3} \text{ ft.}$$

$$OG = \sqrt{OP^2 + GP^2} = \sqrt{121 + 48} = 13 \text{ ft.}$$

Area of triangle $OAB = \frac{1}{2} \,.\, OG \,.\, AB = \frac{1}{2} \,.\, 13 \,.\, 8$ sq. ft.

$= 52$ sq. ft.

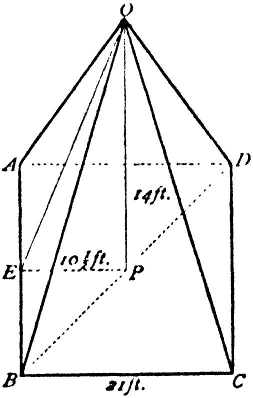
$\therefore$ area of side-faces $= 6 \times 52 = 312$ sq. ft.

And area of hexagonal base $= \frac{3\sqrt{3}}{2} \,.\, 8^2 = 96\sqrt{3}$ sq. ft.

$= 166{\cdot}3$ sq. ft. (nearly).

$\therefore$ total surface $= 478{\cdot}3$ sq. ft. Answer.

(3) It is desired to cover a piece of ground 21 ft. square by a pyramidal tent 14 ft. in perpendicular height. Find the cost of the requisite canvas at 5*d.* a sq. yd.



Let $ABCD$ be the base, and O the vertex. Draw OP = altitude.

Amount of canvas = 4 times triangle $AB = 2AB \times OE$.

In the triangle OEP,

$EP = \frac{1}{2} BC =$ 1(

and $OP = 14$ ft.

$\therefore\ OE$

Amount of canvas

$= 2 \,.\, \times 21 \times \frac{35}{2} = 21 \times 35$ sq. ft.

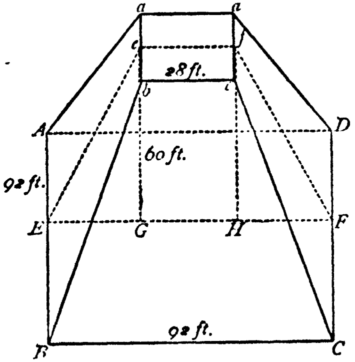
$= \frac{245}{3}$ sq. yds.

$\therefore$ cost of canvas

$= 5 \times \frac{245}{3} d. = £1\ 14s.\ 0\frac{1}{3}d.$ Answer.

(N.B.—We have assumed here that the pyramid is symmetrical. In practical cases, such as tents, church-spires, &c., this assumption may be usually made.)

(4) Find the cost of facing with plaster, at $13\frac{1}{2}d.$ per sq. yd., the sides of an imperfect square pyramid, the sides of whose top and bottom surfaces are 28 ft. and 92 ft., and whose height is 60 ft.



The figure will be a frustum of a pyramid, of which we have to find the lateral surface.

Bisect ab, AB, cd, CD; let a plane pass through the points of bisection. Then $eEFf$ is a trapezoid.

Surface of pyramid

$= 4$ times trapezoid $aABb$.

$= 2\ (ab + AB)\ .\ eE.$

$= 2\ (28 + 92)\ .\ eE.$

$= 240\ .\ eE$ sq. ft.

Also $EG = HF = \frac{1}{2}\ (EF - ef) = \frac{1}{2}\ (92 - 28) = 32$ ft.

$\therefore\ Ee = \sqrt{eG^2 + EG^2} = \sqrt{60^2 + 32^2} = 68$ ft.

$\therefore$ surface of pyramid $= 240 \times 68$ sq. ft.

$\therefore$ cost of plastering $= \dfrac{240 \times 68}{9} \times 13\frac{1}{2}d.$

$= £\dfrac{\overset{34}{\cancel{68}} \times \overset{3}{\cancel{27}}}{\cancel{2} \times \cancel{9}} = £102.$ Answer.

EXAMPLES ON THE PYRAMID

(LENGTHS AND SURFACES)

[N.B.—The expression 'square pyramid' is an abbreviation for 'symmetrical pyramid on a square base.']

1. Find the altitude and total exterior surface of a tetrahedron whose side is 4 ft. 5 in.

2. Find the total exterior surface of a square pyramid, if the side of square is 14 ft., and the altitude 24 ft.

3. The base of a symmetrical triangular pyramid is equilateral (side = 3 in.), and the altitude is 10 in. Find the total exterior surface.

4. The base of a pyramid is a regular pentagon, whose side is 1 in., and the altitude is 2 in. If the vertex is immediately above one of the angular points of the pentagon, find the five edges, and show how the exterior surface may be found.

5. The base of a symmetrical pyramid is a regular hexagon, whose side is 6 in. If the altitude is 3 in., find the lateral surface.

6. The altitude of a symmetrical pyramid is 5 ft. 3 in., and each of the six slant edges is 5 ft. 5 in. Find the area of the base to the nearest sq. in.

7. Find the area of the six equal faces of a hexagonal pyramid, each side of the base being 6 ft., and the perpendicular height of the pyramid being 8 ft. (*Sandhurst.*)

8. A square pyramid has all its edges equal. If its exterior surface is 43.712 sq. in., find its edge, taking $\sqrt{3} = 1.732$.

9. The exterior surface of a tetrahedron is 60 sq. ft. Find its altitude.

10. Find the edge of a tetrahedron which has its whole surface equal to that of a square pyramid, the side of whose base is 6 in., and each of whose other edges is 5 in. Answer to three places of decimals.

11. Find the edge of a cube which has its surface double that of the frustum of a pyramid whose bases are squares containing 49 sq. in. and 81 sq. in. respectively, if the distance between two corresponding sides of the bases is 4 in. Answer to two places of decimals.

12. Find the exterior surface of the frustum of a pyramid, the bases of which are regular hexagons, whose sides are 10 in. and 6 in. respectively, and each of whose other edges is 3⅛ in.

13. In a pyramid on a square base, whose side is 24 ft., and whose altitude is 6 ft., find the length of the perpendicular from the vertex on one of the sides of the base. Find also the side-edge of the pyramid.

14. The bases of the frustum of a pyramid are squares, whose sides are 10 ft. and 20 ft. respectively, and the altitude is 12 ft. Find the whole surface.

15. A square pyramid stands on a base of ⅜ of an acre, and the slope of each side to the plane of the base is 30°. Find the altitude and exterior surface.

16. It is desired to cover a piece of ground 80 ft. square by a pyramidal tent 30 ft. in perpendicular height. Find the cost of the requisite quantity of canvas at 4½*d.* per sq. yd. (*Sandhurst.*)

17. If the area to be covered = 96 ft. square, altitude = 14 ft., and canvas is 5*d.* the sq. yd., find the cost of canvas for pyramidal tent.

18. The cost of a square pyramidal tent covering an area of 5184 sq. ft. is £17 11*s.*, at ¾*d.* per sq. ft. Find the height of the tent.

19. Find the cost of covering with lead a spire, in shape a square pyramid 60 yds. high, the side of whose base is 38 ft., if lead costs 2*s.* 6*d.* per sq. ft.

20. Find the lateral surface of a regular octagonal lantern, if the sides of the bases are 26 ft. and 10 ft., and each of the side-edges is 17 ft.

21. An octagonal spire is 48 ft. perimeter at the base, and 30 yds. high. Find the cost of covering it with lead at 2*s.* 9*d.* the sq. ft.

22. Find the cost of facing with brick, at 5*s.* 6*d.* per sq. yd., the sides of an imperfect square pyramid, the sides of whose top and bottom surfaces are 51 ft. and 93 ft., and whose altitude is 72 ft.

23. A tent is in shape a frustum of a square pyramid, surmounted by another square pyramid. The side of the base is 20 ft., and of the top surface of the frustum 6 ft. The total height is 28 ft., and height of the frustum 24 ft. Find the number of sq. yds. of canvas required to make the tent, and its cost at 4½*d.* per sq. yd.

24. Find the area of each of the sloping surfaces of a frustum of a pyramid whose perpendicular height is 6 in., and which stands on a square base whose side is 6 in., the side of the square top being 1 in. (*Oxford Local.*)

25. Three points, *A*, *B*, and *C*, on the three edges of a cube of wood which meet in *O* are distant 15, 16, and 20 in. respectively from *O*. If the piece *OABC* be sawn off and placed with *O* uppermost on a horizontal table, find the area of the face *ABC*, and the height of *O* above the table. (*Oxford Local.*)

Sec. II.—Volumes

[FORMULÆ:

(1) Volume of *pyramid* $= \frac{1}{3}$ base × altitude.

(2) Volume of *tetrahedron* (edge a) $= \dfrac{a^3}{6\sqrt{2}}$.

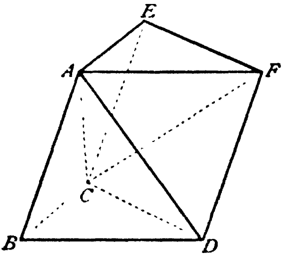
(3) Volume of *frustum of pyramid* $= \dfrac{h}{3}\{B + \sqrt{BB'} + B'\}$ (where $h =$ altitude, and B, B' the bases.)]

84. *Volume of pyramid* $= \dfrac{Bh}{3}$ (where B is the base, and h the altitude).

The proof of this formula is hard and complicated, but the accompanying figure will enable the student to understand partly how the result is obtained.

The formula asserts that *a pyramid is one-third of a prism with the same base and altitude.*

The proof of this depends on the fact that every prism can be divided into three equal pyramids.



Let $ABCDFE$ be a triangular prism.

First let a plane pass through ACD; this will detach a pyramid $ABCD$, whose vertex is A, and base BCD.

Consider this pyramid removed; then the remainder is a pyramid $ACDFE$, on a quadrilateral base $CEFD$, and whose vertex is A.

By letting a plane pass through ACF, this pyramid will be divided into two triangular pyramids, with a common vertex A, and whose bases are CDF and CEF respectively.

Thus the whole prism can be divided into three pyramids; namely, *ABCD*, *ACDF*, *ACFE*.

These three pyramids can be proved to be all equal.*

$\therefore$ volume of pyramid $ABCD = \frac{1}{3}$ prism.

$= \frac{1}{3}$ base $BCD \times$ altitude

(since the altitudes of the prism and pyramid are the same).

When the formula has been proved to hold for a triangular pyramid, it can be readily extended to any pyramid; for every pyramid can be divided by planes into triangular pyramids, as the student will easily see.

Examples.—(1) Find the weight of a pyramidal spire 72 ft. high, on a regular hexagonal base whose side is 5 ft., if 1 cubic ft. of the stone composing it weighs 161 lb.

Area of hexagonal base $= \frac{3\sqrt{3}}{2} \cdot 5^2 = \frac{75\sqrt{3}}{2}$ sq. ft.

$\therefore$ volume of pyramid $= \frac{1}{3} \cdot \frac{75\sqrt{3}}{2} \cdot 72$ cubic ft.

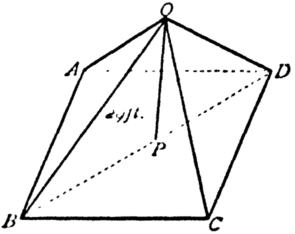
$= 900\sqrt{3}$ cubic ft.

Weight of stone $= 900\sqrt{3} \times 161$ lb.

By working this out, we shall find the weight to the nearest pound $= 250974$ lb.

$= 112$ tons 3 qrs. 10 lb. Answer.

(2) The base of a square pyramid contains 800 sq. ft., and each of the side-edges is 29 ft. Find the volume.



We must find the altitude.

Here $OB = 29$ ft.

Area of base $= 800$ sq. ft.

$\therefore$ side of base $BC = 20\sqrt{2}$ ft.

Diameter $= 20\sqrt{2} \times \sqrt{2} = 40$ ft.

$BP = 20$ ft.

Hence altitude OP

$= \sqrt{OB^2 - BP^2} = \sqrt{29^2 - 20^2}$.

$= 21$ ft.

$\therefore$ volume $= \frac{1}{3} \cdot 800 \times 21 = 5600$ sq. ft. Answer.

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* **The student will observe that the above is no proof of the formula, since the important step in this line is assumed.**

(3) Find the number of cannon-balls which can be piled on a square base with 6 balls in each side.

Consider the successive *layers*.

The bottom row contains $6^2 = 36$ shot.
The next „ „ $5^2 = 25$ „
The third „ „ $4^2 = 16$ „
The fourth „ „ $3^2 = 9$ „
The fifth „ „ $2^2 = 4$ „
The sixth „ „ $1 = 1$ „
$\therefore$ total number of shot $= \overline{91}$ Answer.

This process can be abbreviated. Let n = number of balls in each side of the base; then total number of balls is the sum of the series: $1^2 + 2^2 + 3^2 + \ . \ . \ . \ . \ . + n^2$.

Thus number of balls $= \frac{1}{6} n (n+1)(2n+1)$.

Here $n = 6$, $\therefore$ answer $= \frac{1}{6} \cdot 6 \times 7 \times 13 = 91$ as before.

(4) Find the number of cannon-balls which can be piled on a triangular base with 5 balls in each side.

The bottom row contains $5 + 4 + 3 + 2 + 1 = 15$ balls.
The second „ „ $4 + 3 + 2 + 1 = 10$ „
The third „ „ $3 + 2 + 1 = 6$ „
The fourth „ „ $2 + 1 = 3$ „
The fifth „ „ $1 = 1$ „
$\therefore$ total number of balls $= \overline{35}$ Answer.

Here again it will be most convenient to use a formula. Let n = number of balls in each side of base.

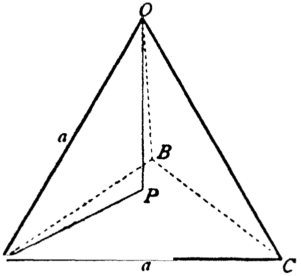
$\therefore$ number of balls $= \frac{1}{6} n (n+1)(n+2)$.

Here $n = 5$, $\therefore$ answer $= \frac{1}{6} \cdot 5 \times 6 \times 7 = 35$ as before.

The two formulæ here used are taken from the summation of series in Algebra.

85. *Volume of tetrahedron* $= \frac{a^3}{6\sqrt{2}}$.

Let $OABC$ be a tetrahedron, every face being an equilateral triangle whose side is a. Draw OP perpendicular to the base; then P is the centre of the circle circumscribed to ABC.



$$\therefore\ AP = \frac{a}{2\sin 60^\circ} = \frac{a}{\sqrt{3}}.$$

Altitude OP

$$= \sqrt{OA^2 - AP^2} = a\sqrt{1-\tfrac{1}{3}}.$$

$$= a\sqrt{\tfrac{2}{3}}.$$

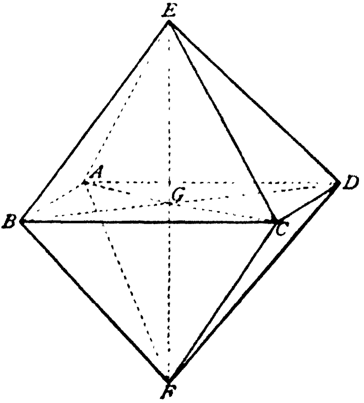
And area of base $= \dfrac{\sqrt{3}\cdot a^2}{4}.$

$$\therefore\ \text{volume of tetrahedron} = \tfrac{1}{3}\cdot\frac{\sqrt{3}\cdot a^2}{4}\cdot\frac{a\sqrt{2}}{\sqrt{3}}.$$

$$= \frac{\sqrt{2}\cdot a^3}{12} = \frac{a^3}{6\sqrt{2}}. \qquad \text{Q.E.D.}$$

86. The tetrahedron is one of the five regular polyhedra, on which see appendix to the present book.

A second is the *cube*, and a third is called the *octahedron*, whose volume may be readily deduced from that of a pyramid.



DEFINITION 15.—An octahedron is a solid figure contained by eight equal and equilateral triangles.

The octahedron may be considered the sum of two pyramids, whose vertices are E and F, and whose common base is $ABCD$. Now, since all the edges and all the solid angles of the solid are equal, it is clear that $ABCD$ is equilateral and equiangular; i.e. it is a square.

If $a =$ side of octahedron, base $= a^2$.

$$\text{and altitude } EG = BG = \frac{a}{\sqrt{2}}.$$

$$\therefore \textit{ volume of octahedron} = \frac{2}{3} \times \frac{a}{\sqrt{2}} \times a^2 = \frac{\sqrt{2} \,.\, a^3}{3}.$$

Examples.—(1) The volume of a tetrahedron is 25 cubic ft. Find its edge to two decimal places of a ft.

Let a be the edge.

$$\therefore \frac{a^3}{6\sqrt{2}} = 25 \text{ cubic ft.}$$

$$a^3 = 150\sqrt{2}.$$

$$= 212{\cdot}13195.$$

∴ extracting the cube root, edge $= 5{\cdot}96$ ft. Answer.

(2) The surface of a tetrahedron is twice that of an octahedron. Compare their volumes.

Let a, a' be their edges.

$\therefore$ surface of tetrahedron $= \sqrt{3} \,.\, a^2$.

" octahedron $= 2\sqrt{3} \,.\, a'^2$;

$\therefore$ $a^2 = 4a'^2$ and $a = 2a'$.

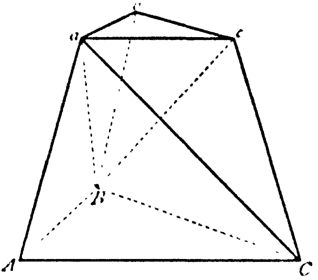
Ratio of their volumes =

$$\frac{a^3}{6\sqrt{2}} : \frac{\sqrt{2}}{3}a'^3 = \frac{8a'^3}{6\sqrt{2}} : \frac{\sqrt{2}}{3}a'^3 = 2 : 1. \quad \text{Answer.}$$

87. Volume of *frustum of pyramid* $= \frac{h}{3}\{B + \sqrt{BB'} + B'\}$.

Every frustum of a pyramid can be divided by planes into three pyramids, the sum of which is the volume of the frustum.

This can be proved in the case of a triangular pyramid by a figure like that used to illustrate the volume.



In the frustum let the base ABC be denoted by B, and abc by B'.

First let a plane pass through aBC, this will detach a pyramid with base ABC and vertex a, whose volume $=\frac{hB}{3}$.

Suppose this pyramid removed, the remainder is a pyramid whose vertex is a, on a quadrilateral base $BCcb$.

This can be divided into two pyramids $aBcb$ and $aBCc$. The former may be considered as on a base abc, and with vertex B. $\therefore$ its volume $=\frac{hB'}{3}$.

The remaining pyramid can be shown $=\frac{h}{3}\sqrt{BB'}$. Of course this assumes, as before, the hardest part of the proof, but it will serve as an illustration.

Example.—The sides of the base of the frustum of a triangular pyramid are 2 ft. 4 in., 2 ft. 1 in., and 1 ft. 5 in., and the longest side of the top surface is 4 in. If the altitude is 1 ft. 9 in., find the volume.

Area of base $=\sqrt{\underset{5}{\cancel{35}}\times\cancel{7}\times\cancel{10}\times\underset{9}{\cancel{18}}}=7\times 10\times 3=210$ sq. in.

Now the top surface is similar to the base or bottom surface.

$\therefore$ top surface : base :: $4^2 : 28^2$.

:: $1 : 49$.

$\therefore$ top surface $=\frac{1}{49}\times 210=\frac{30}{7}$ sq. in.

$\therefore$ $B=210$ sq. in., $B'=\frac{30}{7}$ sq. in., $h=21$ in.

$$\text{Volume}=\frac{21}{3}\left\{210+\sqrt{210\times\frac{30}{7}}+\frac{30}{7}\right\}=7\left\{210+30+\frac{30}{7}\right\}$$

$$=1680+30=1710 \text{ cubic in. Answer.}$$

EXAMPLES ON THE VOLUME OF THE PYRAMID

1. Find the volume of a pyramid whose base is 10 sq. poles in area, and whose altitude is 78 yds. 2 ft. 4 in.

2. Find the volume of a triangular pyramid, the sides of the base being 26 ft. 3 in., 22 ft. 9 in., and 24 ft. 6 in., and the altitude being 25 ft.

3. Find the volume of a square pyramid, if side of the base is 10 ft., and each of the side-edges is 20 ft.

4. A symmetrical pyramid has a regular pentagon for its base, each side being 24 ft. If the altitude is 15 ft., find the cubical content of the pyramid to the nearest cubic ft.

5. In a square pyramid, given altitude = 17 ft. 3 in., and diagonal of base = 10 ft. 4 in., find volume.

6. In a square pyramid, given that the edge = 5 ft. 10 in., and the distance from vertex to middle point of side of base = 4 ft. 2 in., find volume.

7. In a square pyramid, given that side of base = 2 ft. 4 in., and the distance from vertex to middle point of side of base = 4 ft. 2 in., find the volume.

8. In a square pyramid, given that volume = 9 cubic ft. 832 in., and altitude = 8 in., find (1) area of base, (2) edge, (3) distance from vertex to middle point of side of base.

9. If the volume of a pyramid is 34 cubic yds. 22 ft., and the base is 23 sq. yds. 4 ft. 72 in., find the altitude.

10. The base of a pyramid covers an area of $13\frac{2177}{8712}$ acres, and the height is 480 ft. Find the side of the square, and the volume of the pyramid. (*Sandhurst.*)

11. If the altitude of a hexagonal pyramid is 8 ft., and each side of the base is 6 ft., find the cubical content of the pyramid.

How must a plane be drawn parallel to the base so as to divide the pyramid into two parts whose contents shall be equal to one another? (*Sandhurst.*)

12. A right pyramid, whose base is a square of 7 in. the side, and whose perpendicular height is 8 in., is cut into two parts by a plane parallel to the base, and 6 in. from it. Find the volume of the two parts, and their total surface. (*Sandhurst.*)

13. Find the volume of a tetrahedron whose surface is 50 sq. in. Given log 2 = .30103, log 3 = .47712, log 18279 = 4.26195.

14. Find the volume of an octahedron whose edge is 3 in., to three places of decimals.

15. A square pyramid has all its edges equal, and also equal to those of a tetrahedron. Show that the volume of the pyramid is double that of the tetrahedron.

16. Find the volume of an octahedron whose surface is 36 sq. in. Given log 2 = .30103, log 3 = .47712, log 15793 = 4.19846.

17. Find the volume of an octahedron whose surface is 8 times that of a tetrahedron whose edge is 2 ft. 6 in. Answer to two decimal places of a cubic ft.

18. If the edges of a tetrahedron and an octahedron are equal, the octahedron is 4 times as large as the tetrahedron.

19. Find the volume of the frustum of a triangular pyramid, if the altitude of frustum is 2 ft. 6 in., the sides of base 25 in., 42½ in., and 52½ in., and smallest side of upper surface 10 in.

20. Find the volume of the frustum of a pyramid whose bases are 42 sq. ft. and 168 sq. ft., and altitude 20 ft.

21. Find the volume of the frustum of a square pyramid, if the side of the base is 18 ft., of the top surface 11 ft., and each of the slant edges is 12½ ft. Answer correct to a ft.

22. The ends of the frustum of a pyramid are rectangles whose areas are in the proportion of 4 to 9. The sides of the bottom rectangle are 8 yds. and 7 yds., and the altitude is 16 yds. Find the volume.

23. The sides of the ends of the frustum of a tetrahedron are 4 in. and 1 in. respectively. Find the altitude and the volume of the frustum, each to three decimal places.

24. Find the weight of the frustum of a square stone pyramid, the sides of whose ends are 7 ft. 3 in. and 5 ft. 9 in., and whose altitude is 5 ft. 4 in., if the stone weighs 39 cwt. 3 qrs. 3 lb. per cubic yd.

25. The side of a tetrahedron is 12 ft. Find correct to a shilling the cost of the stone composing it, at £1 3*s.* 4*d.* per ton, if every cubic ft. weighs 1½ cwt.

26. Find the weight of the stone in a church-spire on a square base, if the side of the base is 8 ft., and the height is 50 ft., given that 1000 oz. of stone contain ⅜ cubic ft.

27. A pyramidal spire on a square base of 128 sq. yds. is to be placed on a church-tower. If the tower will stand a weight of 2880 tons, and 1 cubic ft. of stone weighs 160 lb., find the greatest possible height to which the spire may be carried.

28. A pyramidal spire 84 ft. high, on a square base, is made of stone, the total weight of which is 1436¼ tons. Find the cost of covering it with lead at 2*s.* 7½*d.* the sq. yd., given that a cubic ft. of stone weighs 170 lb.

29. The great pyramid of Egypt was 481 ft. in height when complete, and its base was 764 ft. in length. Find the volume to the nearest number of cubic yds. (*Sandhurst.*)

30. Cleopatra's Needle consists approximately of a frustum of a pyramid surmounted by a smaller pyramid. If the lower base was 7½ ft. square, and the upper one 4½ ft. square, the height of the frustum being 61 ft., and of the upper pyramid 7½ ft., find its cubical content and weight, if 1 cubic ft. weighs 170 lb. (*Sandhurst.*)

31. A cup is in shape an inverted frustum of a hexagonal pyramid, the sides of whose bases (internal dimensions) are $2\frac{1}{2}$ in. and $1\frac{2}{3}$ in. If $1\frac{2}{3}$ pints of water poured into the cup fills $\frac{7}{8}$ of it, find the depth of the cup, exact to $\frac{1}{10}$ of an in.

32. The upper part of a funnel is the inverted frustum of a square pyramid. If the sides of the top and bottom squares are 4 in. and $\frac{1}{8}$ in. respectively, and the height is 9 in., find the number of pints it contains, if 1 pint contains 34.66 cubic in. Answer to $\frac{1}{100}$ of a pint.

33. Find the number of cannon-balls that can be arranged in a pyramidal heap on a square base, there being 10 balls in the side of the lowest layer. If the diameter of a ball is 6 in., find the size of a pyramidal box that will just cover them ($\sqrt{2} = \frac{7}{5}$, $\sqrt{3} = \frac{26}{15}$). (*Sandhurst.*)

34. Find the number of cannon-balls in a pile on a triangular base containing 7 balls in each side.

35. Find the number of cannon-balls in a pile on a square base containing 12 balls in a side.

36. How many more balls can be piled on a square base than on a triangular base, the number of rows in each being 15?

37. Find the number of cannon-balls in an incomplete triangular pile, whose top layer has 7 balls in a side, and bottom layer 15 balls in a side.

38. Find the number of cannon-balls in a frustum of a square pile, whose top and bottom surfaces have 14 and 20 balls in their sides respectively.

39. Find the number of cannon-balls in a triangular pile, if the bottom layer contains 45 balls in all.

40. If the number of cannon-balls in a triangular pile is to the number of cannon-balls in a square pile with the same number of rows as 14 : 25, find the number of rows.

CHAPTER V.—THE CYLINDER

Section I

[FORMULÆ:

(1) Lateral surface of *cylinder*

$$= \text{circumference of base} \times \text{altitude}.$$

(2) Lateral surface of *frustum of cylinder*

$$= \text{circumference of base} \times \frac{h_1 + h_2}{2}$$

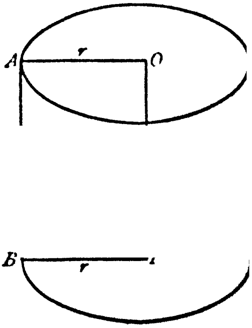
(where h_1, h_2 are the longest and shortest edges.)]

88. DEFINITION 16.—"A cylinder is a solid figure described by the revolution of a right-angled parallelogram about one of its sides which remains fixed."

"The axis of a cylinder is the fixed straight line about which the parallelogram revolves."

"The bases of a cylinder are the circles described by the two revolving opposite sides of the parallelogram." (Euclid, xi. def. 21-23.)

For example, suppose the side OP of the rectangle $OABP$ to remain fixed, and the side AB to revolve round it. Then AB, as it revolves, traces out the surface of a cylinder.



A and B trace out equal circles, which are called the *bases*, and OP is called the *axis*. The axis is clearly coincident with the length, which is the same as the altitude.

The cylinder is a common figure in practice. A ruler, a stone roller, a round tower, are familiar examples.

89. The student who has learnt the use of the term 'locus' can form another conception of a cylinder. A cylinder is the locus of a straight line which moves parallel to a fixed straight line, and at a fixed distance from it. Or we may call it the locus of a point whose perpendicular distance from a fixed straight line is always the same.

90. We have defined a cylinder in the only sense in which we shall use the term. But it is necessary to observe that mathematicians use the term in a far wider sense. Our definition is really that of a *right circular cylinder.*

It is called *right* because its axis is perpendicular to its base.

It is called *circular* because its bases are circles.

But there are cylinders whose axes are inclined to their respective bases at an oblique angle, and also cylinders whose bases are other curves than a circle.

Extended Definition of a Cylinder.—A cylinder is the solid contained by a straight line which always moves parallel to itself, and passes through some point on a given curve.

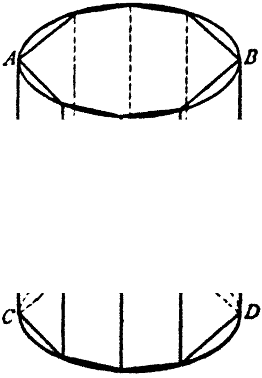
We shall not refer to this extended definition again.

91. *Connection between prism and cylinder.*

Let $ABCD$ be a cylinder. In each of the bases inscribe a regular polygon of the same number of sides, and join the corresponding angular points. Then we have a *regular polygonal prism* inscribed in the cylinder.

By increasing the number of sides, each of the bases of the prism can be made to differ by as little as we please from the bases of the cylinder. (See Art. 34.)

It follows that the prism, if we increase its number of side-faces, can be made to differ by as little as we please from the cylinder, both in surface and volume.



Hence the cylinder may be regarded the limit of a prism, and all its properties deduced from it.

92. *Surface of a cylinder.*

The surface of a cylinder consists of two parts; i.e. the bases, which are plane circles, and the curved surface of the sides, which is called the *lateral surface.*

If we cut a rectangle out of paper, and bend round two sides so as to meet, we shall practically have a cylinder. This will show that the lateral surface of a cylinder may be conceived as unrolled and laid flat, when it will form a rectangle, of which the length is the circumference of the base, and the breadth the altitude of the cylinder.

Hence, if h be the altitude, and r the radius of the base,

$$\textit{Lateral surface of cylinder} = 2\pi r h.$$

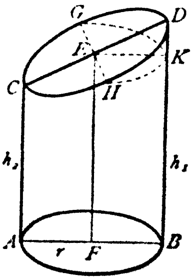
If the *whole* exterior surface is required, we must add the ends, each of which is a circle with radius r.

$\therefore$ total surface of cylinder $= 2\pi r h + 2\pi r^2 = 2\pi r\,(h + r)$.

93. *Frustum of cylinder.*

DEFINITION 17.—A frustum of a cylinder is that part cut off by a plane not parallel to its base.

For instance, the figure shows the frustum of a cylinder cut by a plane whose section with the lateral surface is a curve $CGDH$. If the cutting-plane were parallel to the base, its section with the lateral surface would be a circle equal to the base. Hence a cylinder will be cut, by a plane parallel to the base, into two cylinders with equal bases, and differing only in height. The student should compare this with the parallel case of the prism.



When the cutting-plane is not parallel with the base, its section with the lateral surface is not a circle at all, but a curve called an ellipse.

94. *Lateral surface of frustum of cylinder.*

Let C and D be the highest and lowest points where the cutting-plane meets the lateral surface of the cylinder. Then BD and AC may be called the *longest and shortest edges.* Let them be h_1 and h_2.

Let E be the point where the cutting-plane meets the axis of the cylinder. Through E draw a plane GKH parallel to the base. This will cut off from the top of the frustum a slice $GHKD$ which will exactly fit on to the other side, completing a cylinder whose altitude is EF.

Thus the frustum = a cylinder with base AB and altitude EF.

Now $EF = \frac{1}{2}(AC + BD) = \frac{h_1 + h_2}{2}$.

$$\therefore \text{ lateral surface of frustum} = 2\pi r \times \frac{h_1 + h_2}{2}.$$

$$= \pi r(h_1 + h_2).$$

Examples.—(1) Find the cost of plastering inside, at 1*s.* 2*d.* per sq. yd., the walls of a round tower, whose extreme (internal) breadth is 12 ft., to the height of 30 ft.

Area of walls = circumference × height.

= 12π × 30 sq. ft. = 40π sq. yds.

$\therefore$ cost of plastering = $14 \times 40d. \times \frac{22}{7} = 44 \times 40 = 1760d.$

= £7 6*s.* 8*d.* Answer.

(Notice that, as the circumference is the perimeter of the tower, we have once more the formula:

Area of walls = perimeter × height.)

(2) A garden-roller is 2 ft. 11 in. in diameter, and 3 ft. long. How many times will it turn round in rolling 4 acres of land?

Each time the roller turns round it will roll a piece of ground exactly equal in area to its lateral surface.

Lateral surface of roller $= 35\pi \times 36$ sq. in.

$$= \overset{5}{\cancel{35}} \cdot \frac{\overset{11}{\cancel{22}}}{\cancel{7}} \cdot \cancel{36} \cdot \frac{1}{\underset{\underset{2}{\cancel{4}}}{\cancel{144}}} \text{ sq. ft.}$$

$$= \tfrac{55}{2} \text{ sq. ft.}$$

Number of sq. ft. in 4 acres $= 4 \times 4840 \times 9$.

$\therefore$ number of times roller will turn round

$$= 4 \times \overset{88}{\cancel{4840}} \times 9 \times \frac{2}{\cancel{55}} = 6336. \quad \text{Answer.}$$

(We can only rely on the correctness of three figures in the answer. If we calculate more exactly we shall find the number of times $= 6338 \cdot 5$, &c. This example well illustrates the assertion made above, that a cylinder, if *unrolled*, becomes a rectangle.)

(3) Find the lateral surface of the frustum of a cylinder whose longest and shortest edges are 3 ft. 7 in. and 2 ft. 5 in., and the diameter of whose base is 4 ft. Answer correct to a sq. in.

Here $h_1 = 3$ ft. 7 in., $h_2 = 2$ ft. 5 in., $\therefore \dfrac{h_1 + h_2}{2} = 3$ ft.

$\therefore$ lateral surface $= 3 \times 4\pi$ sq. ft.

$= 12\pi$ sq. ft.

The number of sq. in. in the answer will obviously be greater than 1000, so that, to insure the answer being correct to a sq. in., we shall have to take $\pi = 3 \cdot 1416$.

$\therefore$ lateral surface $= 37 \cdot 6992$ sq. ft.

$= 37$ sq. ft. $100 \cdot 7$ sq. in.

$\therefore$ surface correct to a sq. in. $= 37$ sq. ft. 101 sq. in.

EXAMPLES ON THE CYLINDER

(LENGTHS AND SURFACES)

(N.B.—π, except where otherwise stated, $= \frac{22}{7}$. When this value is given for π, it must be noticed that the answers are often given exactly, though really only approximately correct.)

1. Find the lateral surface of a cylinder whose altitude is 5 ft. 10 in., and the radius of whose base is 2 ft. 3 in.

2. Find the whole surface of a cylinder whose altitude is 8 ft., and diameter of whose base is 4 ft. 6 in.

3. Find the curved surface of a cylinder 10.438 in. long, if the circumference of the base is 8.295 in.

4. The base of a cylinder is $38\frac{1}{2}$ sq. ft. in area, and the axis is 15 ft. Find the total surface.

5. The curved surface of a cylinder is $1005\frac{5}{7}$ sq. ft., and the whole surface is 1408 sq. ft. Find the length.

6. The curved surface of a cylinder, whose altitude is 25 ft., is 2200 sq. ft. Find the whole surface.

7. The curved surface $= 26\frac{1}{8}$ sq. ft., and the altitude $= 4$ ft. 9 in. Find the radius of the base.

8. The whole surface $= 94\frac{2}{7}$ sq. ft., and the diameter of the base $= 4$ ft. Find the altitude.

9. The curved surface $= 100$ sq. ft., and the radius of the base $= 5$ ft. Find the length.

10. A cylinder is formed by the revolution of a rectangle, whose sides are 3 in. and 5 in. round its shortest side. Find its lateral surface. What would be the lateral surface if the rectangle revolved round the longer side?

11. If the altitude of a cylinder = radius of base, the lateral surface = $\frac{1}{2}$ whole surface.

For instance, if the lateral surface = 136 sq. ft. 128 in., find the diameter of base.

12. Find the whole amount of surface on a hollow cylinder whose height is 1 ft., and whose internal and external radii of base are 4 in. and 6 in. (*Sandhurst.*)

13. Find the lateral surface of the frustum of a cylinder whose longest and shortest edges are 7 in. and 5 in., and the diameter of whose base is 4 in.

14. Find the lateral surface of the frustum of a cylinder, if the radius of the base is 15 yds. 1 ft., and the sum of the longest and shortest edges is 32 yds. $1\frac{1}{2}$ ft.

15. The lateral surface of the frustum of a cylinder is 238 sq. ft., and the longest and shortest edges are 11 ft. 5 in. and 5 ft. 7 in. Find the area of the base.

16. The diameter of the base of a cylinder is 6 ft. Find the length of a piece cut off by a plane parallel to the base whose lateral surface = that of the frustum of a cylinder on a base with a diameter three times as large, and the longest and shortest edges of which are 3 ft. 4 in. and 4 ft. 8 in.

17. A garden-roller, 3 ft. 6 in. in diameter, and 3 ft. 8 in. long, makes 10 revolutions in passing from one end of a lawn to another. Find the area rolled in 14 journeys up and 13 journeys down.

18. Find the interior surface of a circular well 42 yds. deep and 3 ft. 8 in. broad at the mouth.

19. Find the cost of painting the walls of a circular tower at 2*s.* 4*d.* per sq. yd., if the diameter is 4 yds. and the height 60 ft.

20. If the breadth of a circular room is 16 ft., and the cost of painting the walls at 2*s.* $7\frac{1}{2}$*d.* the sq. yd. comes to £6 12*s.*, find the height.

21. A garden-roller 2 ft. 11 in. in diameter, and 4 ft. 7 in. wide, is drawn across a lawn 5 chains by 2½ chains at the rate of 8 revolutions a minute. Find the time it will take to roll the whole lawn.

22. Find the number of sq. yds. of sheet-iron required to make 252 yds. of piping with a diameter of 8 in.

23. Find the number of sq. in. of glass in a cylindrical glass tumbler 4 in. deep and 2½ in. broad. ($\pi = 3.1416$.)

24. Find the exterior surface of a round tower with a flat roof 70 ft. high and 10 yds. in external diameter, and the cost of plastering the whole at 1½*d.* per sq. ft. Answer correct to a sq. yd.

25. Find the number of sq. ft. of paint on 1000 circular pencils, each 6 in. long and $\frac{3}{16}$ in. broad. ($\pi = 3.1416$.)

Section II.—Volumes

[FORMULÆ :

(1) Volume of *cylinder* = base × altitude.

(2) Volume of *frustum of cylinder* = base × $\frac{h_1 + h_2}{2}$.]

95. *Volume of cylinder* $= \pi r^2 h$ (where h = altitude, r = radius of base).

The reason for this formula will be readily understood by the student who has grasped the fact that the cylinder is the limit of a regular polygonal prism, when the number of side-faces is indefinitely increased.

For the volume of prism = base × altitude, and the base of the prism ultimately coincides with the base of the cylinder.

∴ volume of cylinder = base × altitude.

Examples.—The practical applications of the cylinder are more numerous than those of any other solid figure excepting, perhaps, the parallelepiped. It will be necessary only to furnish illustrations of a few of them. The three following examples are taken from Sandhurst papers.

(1) If 30 cubic in. of gunpowder weigh 1 lb., what weight of gunpowder will be required to fill a cylinder of 8 in. internal diameter, and with a length of 2½ ft.?

Volume of cylinder $= \pi \cdot 4^2 \cdot 30$ cubic in.

$$\therefore \text{ number of lbs. of gunpowder} = \frac{\pi \cdot 16 \cdot 30}{30} = 16\pi.$$

If we take $\pi = \frac{22}{7}$ we obtain

weight of gunpowder = 50 lb. 4 oz. Answer.

This result is correct to an ounce.

(2) Water is poured into a cylindrical reservoir, 20 ft. in diameter, at the rate of 400 gallons a minute. Assuming a gallon of water to measure 277¼ cubic in., find the rate at which the surface of the water rises in the reservoir. (*Sandhurst.*)

Amount of water poured in each minute $= 400$ gallons.
Volume " " $= 400 \times 277\frac{1}{4}$ cubic in.
Also surface of reservoir $= \pi \,.\, 10^2 = 1000\pi$ sq. ft.
$= 14400\pi$ sq. in.

The water poured in will take the form of a cylinder, of which we have to find the *altitude*.

∴ number of in. that water rises every minute

$$= \frac{400 \times 277\frac{1}{4}}{14400\pi} = \frac{110900}{14430\pi} = \frac{1109}{144\pi}.$$

If we take $\pi = \frac{22}{7}$,

∴ number of in. $= \frac{7763}{3168} = 2{\cdot}45$ in. per minute. Answer.

(3) How many cubic in. of iron are there in a garden-roller which is half an inch thick, with an outer circumference of $5\frac{1}{2}$ feet, and a width of $3\frac{1}{2}$ ft.? $(\pi = \frac{22}{7}.)$

The roller is in shape a *hollow cylinder*, and its volume is found by subtracting from what would be its volume if solid throughout, the volume of the unoccupied space within, which is also a cylinder.

Outer circumference $(2\pi r) = \frac{11}{2}$ ft.

∴ outer radius $= \frac{11}{2} \times \frac{7}{44} = \frac{7}{8}$ ft. $= 10\frac{1}{2}$ in.

∴ inner radius $= 10\frac{1}{2} - \frac{1}{2} = 10$ in.

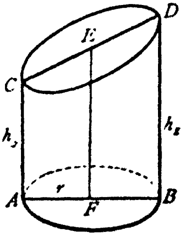
Volume of iron $= \left\{\pi\,(10\frac{1}{2})^2 - \pi \,.\, 10^2\right\}\, 42$ cubic in.

$= 42\pi \,.\, \frac{1}{2} \,.\, 20\frac{1}{2} = 42 \,.\, \frac{22}{7} \,.\, \frac{1}{2} \,.\, \frac{41}{2} = 33 \times 41.$

$= 1353$ cubic in. Answer.

This principle of the *hollow cylinder* is of great use.

96. *Volume of frustum of cylinder* $= \pi r^2 \left(\frac{h_1 + h_2}{2}\right)$ (where h_1, h_2 are the longest and shortest edges).



For it has been proved that the frustum = a cylinder whose base is AB, and whose altitude $= EF$,

∴ volume of frustum = base × EF.

$$= \pi r^2 \left(\frac{h_1 + h_2}{2}\right).$$

Example.—(1) Find the weight in lbs. of a marble column, in shape a frustum of a cylinder, whose longest and shortest edges are 12 ft. 9 in. and 11 ft. 1 in., and whose radius of base is 1 ft. 3 in., if 1 cubic ft. of marble weighs 2716 oz.

$$\text{Volume of marble} = \frac{12\frac{3}{4} + 11\frac{1}{12}}{2} \cdot \pi\left(\tfrac{5}{4}\right)^2 \text{ cubic ft.}$$

$$= \tfrac{143}{12} \cdot \tfrac{25}{16} \cdot \pi.$$

$$\therefore \text{ weight of marble} = \tfrac{143}{12} \cdot \tfrac{25}{16} \cdot \pi \cdot \tfrac{2716}{16} \text{ lb.}$$

If we take $\pi = \frac{22}{7}$, weight $= \frac{143}{12} \cdot \frac{25}{16} \cdot \frac{22}{7} \cdot \frac{2716}{16}$ lb.

This reduces to $\frac{3814525}{384} = 9934$ lb. to nearest lb. Answer.

If we take $\pi = 3{\cdot}1416$, we shall find the exact number of lbs. to be 9930.

EXAMPLES ON THE VOLUME OF THE CYLINDER

1. Find the volume of a cylinder, if the altitude is 8 ft. 5 in., and diameter of base 5 ft. 10 in.

2. Find the volume of a cylinder, if the axis is 30 yds. long, and 12 yds. distant from every point of the curved surface.

3. Given the volume of a cylinder = 114 cubic yds. 2 ft., and the diameter of base = 14 ft., find the altitude.

4. If the volume of a cylinder is 23 cubic yds. 4 ft. 1080 in., and the radius of base is 9 ft., find the lateral surface.

5. If the total surface of a cylinder is 17,600 sq. in., and the height = 3 times the diameter of base, find the volume.

6. Find the volume of a frustum of a cylinder whose longest and shortest edges are 4 ft. 2 in. and 3 ft. 10 in., and the diameter of whose base is 3 ft. ($\pi = 3.1416$.) Answer to four decimal places of a cubic ft.

7. The volume of the frustum of a cylinder is 56 cubic ft. 32 in., and the part of the axis intercepted is 6 ft. 5 in. Find the diameter of the base.

8. A rectangle 2 ft. 5 in. long, and 1 ft. 8 in. broad, is made to revolve round its shorter end. Find the volume of the cylinder it traces out.

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9. Find the number of cubic ft. of air in a circular room 8 ft. 6 in. high, whose extreme breadth is 3 yds.

10. Find to the nearest lb. the weight of a solid round bar of iron 3 ft. long, and 2½ in. in extreme thickness, if 1 cubic in. of iron weighs 4½ oz.

11. Ten cubic ft. of brass are drawn out into wire $\frac{1}{32}$ in. in diameter. Find the length of the wire to the nearest yd. ($\pi = 3.1416$.)

12. Fourteen pounds' weight of iron is drawn out into wire $\frac{1}{12}$ in. in diameter. Find its length in inches to the nearest inch, if a cubic in. of iron weighs 4½ oz. ($\pi = 3.1416$.)

13. A row of 8 round pillars (diameter = 15 in.) are cut off at the top by a sloping roof, which meets them at 18 ft. 3 in. from the ground, and leaves them again at 21 ft. 6 in. from the ground. Find the weight of the pillars, if 6 cubic ft. of the stone of which they are composed weigh 1000 lb.; and their cost at 15*d.* per cwt.

14. The top part of a round wooden table contains 4 cubic ft. 480 cubic in.; the thickness is 3 in. Find the cost of polishing its top surface at 4½*d.* per 100 sq. in.

15. The diameter of a well is 3 ft. 9 in. Find how many gallons of water are in it, if the well is 10 ft. deep.

16. The diameter of a well is 3 ft. 10 in. How many times must a cylindrical bucket 1 ft. deep, and 7⅔ in. in diameter, be drawn up full, so that the surface of the water may be lowered 18 in.?

17. The diameter of a well is 3 ft. 6 in., and its depth is 54 ft. Find the cost of excavation at 15*d.* per cubic yd.

18. The diameter of a cylindrical reservoir is 120 ft. How many gallons of water must be pumped out per hour, so as to lower the surface 6 in. in 12 hours?

19. What weight of water is there in 1½ miles' length of cylindrical pipe whose inner diameter is 7 in.?

20. A circular measure is 21 in. in diameter, and holds 2½ bushels of seed. Find its depth, if 1 gallon contains 277¼ cubic in.

21. A cylindrical pipe 14 ft. long contains 396 cubic ft. Find its diameter, and the cost of gilding its surface at 9¾*d.* per sq. ft. (*Sandhurst.*)

22. A right cylinder, open at the top, with a diameter of 24 in., weighs 167.5 lb. When filled with water it weighs 2131 lb. Find the height of the cylinder, it being given that a cubic ft. of water weighs 62.5 lb. (*Sandhurst.*)

23. A round wax candle is ⅞ in. in diameter, and 5 in. long. If 1 cubic ft. of wax weigh 65 lb., find how many of these candles go to the pound.

24. A wax candle 8 in. long, and whose diameter is 1 in., is cut into two unequal pieces, the extreme length of one being 6 in., and of the other 4½ in. Find the weight of each part, wax weighing as before.

25. Find how many hundred gallons flow in 5 hours through a pipe, whose diameter is 3 in., at the rate of 10½ miles an hour.

26. 100,000 gallons of water flow in 5 hours through a pipe 4 in. in diameter. Find in miles the velocity per hour of the issuing water.

27. Water is poured into a cylindrical reservoir, 24 ft. in diameter, at the rate of 25 gallons per minute. Find the rate at which the surface of the water rises in the reservoir.

28. Water flows in at the rate of 8 miles an hour through a cylindrical pipe 1 ft. in diameter into a cylindrical reservoir, the diameter of which is 140 ft. Calculate the time in which the surface of the water will be raised 1 in.

29. A shilling, containing 22 parts out of 24 pure silver, may be considered ⅞ in. broad and ¼ in. thick. Find the value of a round bar of silver 4 ft. long and 3½ in. thick.

30. A round tin canister holds 7 lb. of a substance, $1\frac{1}{32}$ cubic in. of which go to the oz The height of the canister is 1 ft. Find its exterior lateral surface, neglecting the thickness of the tin.

31. A round canister full of gunpowder weighs 3½ lb. Its total height is 8¼ in., and exterior diameter of base 4¼ in. If 1 lb. of powder occupies 32 cubic in., and the uniform thickness of the material of which the canister is made is ¼ in., find to the nearest oz. the weight of 1 cubic ft. of this material.

32. The greater diameter of a hollow iron roller is 1 ft. 9 in., the thickness of the metal 1½ in., and the length of the roller 5 ft. Supposing a cubic ft. of cast iron to

weigh 464 lb., what would the roller cost at 16*s.* per cwt.? and how many times would it turn round in rolling 5 acres of land? ($\pi = 3.1416$.) (*Sandhurst.*)

33. Find the weight to the nearest lb. of a hollow iron garden-roller, if the extreme diameter is 23 in., thickness of iron $\frac{1}{2}$ in., and width of roller 30 in., 450 lb. of iron going to 1 cubic foot.

34. Find the weight of a hollow iron tube, if the diameter of outer surface is 4 in., thickness of iron $\frac{1}{2}$ in., and length 6 ft., if 240 cubic in. of iron = 1 cubic ft. of water in weight.

35. What is the weight of a cylinder formed of sheet iron $\frac{1}{2}$ in. thick, with an outer circumference of 10 ft. $7\frac{2}{7}$ in., and a width of 3 ft. 6 in.? 240 cubic in. of iron weigh 1000 oz. avoirdupois. (*Sandhurst.*)

36. Find the cost of the stone in a round tower 80 ft. high, the external and internal diameters of which are 34 ft. and 20 ft., if 1 cubic ft. of stone costs 2*s.*

37. Find the amount of wood used in making 4000 lead pencils, each $\frac{3}{16}$ in. in diameter and 6 in. long, if the diameter of the lead in each is $\frac{1}{16}$ in.

38. Find the cost of the bricks for making a well, at 21*s.* per thousand, if the inner diameter of the well is 4 ft. 9 in., thickness of bricks $4\frac{1}{2}$ in., and depth of well 45 ft., the dimensions of each brick being 9 in., $4\frac{1}{2}$ in., and 3 in.

39. A well 5 ft. in diameter and 30 ft. deep is to have a lining of bricks, fitting close together without mortar, 9 in. thick. Required approximately in tons the weight of the bricks, supposing a brick $9 \times 4\frac{1}{2} \times 3$ in. to weigh 5 lb. (*Sandhurst.*)

40. The outer wall of a circular stone tower 108 ft. high is 3 ft. thick, and the inner diameter is 8 ft., a winding stone staircase is exactly built round a central column 1 ft. in diameter, each step has for its top surface part of the sector of a circle, with a bounding arc of 6 ft. Both the column and the steps are of stone. Find the number of cubic yds. of stone in the tower.

CHAPTER VI.—THE CONE

Section I.

[FORMULÆ:

(1) Lateral surface of *cone* = $\frac{1}{2}$ circumference of base × slant side ($\pi r l$).

(2) Lateral surface of *frustum of cone* = $\frac{1}{2}$ sum of circumferences of bases × slant side ($\pi(r_1 + r_2) l$).]

97. DEFINITION 18.—"A cone is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which remains fixed."

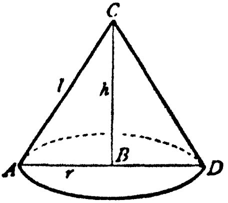
"If the fixed side be equal to the other side containing the right angle, the cone is called a right-angled cone; if it be less than the other side, an obtuse-angled cone; and if greater, an acute-angled cone."

"The axis of a cone is the fixed straight line about which the triangle revolves."

"The base of a cone is the circle described by that side containing the right angle which revolves." (Euclid, xi. defs. 18–20.)

For example, let *ABC* be a right-angled triangle, having a right angle at *B*. Suppose *BC* to remain fixed, and the triangle to revolve round it. Then the side *AC*, as it revolves, traces out the surface of a cone.

BC is called the *axis;* and *BA* describes a circle which is called the *base.*



Also *C* is called the *vertex;* and any straight line drawn from the vertex to any point in the circumference of the base is called the *slant side.* It is obvious that all such straight lines are equal.

The *altitude* is obviously coincident with the axis.

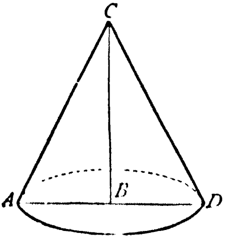
If h be the altitude, l the slant side, and r the radius of the base, then $l^2 = h^2 + r^2$.

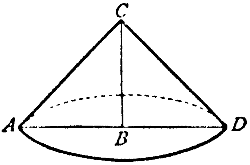
The ordinary extinguisher-shape will serve as an illustration of a cone. Barrows and spires are often conical.

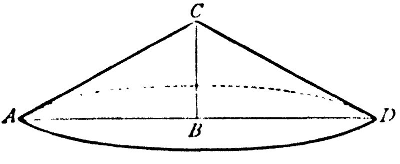
98. *Vertical angle.*

The division of cones into acute-angled, right-angled, and obtuse-angled is according to the *vertical angle.*

Let a plane pass through C the vertex, and AD the diameter of the base of any cone. Then ACD is called the vertical angle.







Since the triangles ABC, DBC are equal, it is clear that BC bisects the vertical angle.

(i) If $AB < BC$, $\therefore$ angle $ACB <$ angle BAC.

$\therefore$ angle ACB is $< 45°$.

$\therefore$ the vertical angle is $< 90°$, and the cone is *acute-angled.*

(ii) If $AB = BC$, $\therefore$ angle $ACB =$ angle BAC.

$\therefore$ each angle $= 45°$.

$\therefore$ the vertical angle $= 90°$, and the cone is *right-angled.*

(iii) If $AB > BC$, then angle $ACB >$ angle BAC.

$\therefore$ angle $ACB > 45°$.

$\therefore$ the vertical angle is $> 90°$, and the cone is *obtuse-angled*.

99. A cone may also be considered as the locus of a straight line which always passes through a given point (the vertex), and always is inclined at a fixed angle to a fixed straight line (the axis).

100. We have defined a cone in the only sense in which we shall use the word. But mathematicians use the word in a far wider sense. Our definition is really that of a *right circular cone.*

It is called *right* because its axis is perpendicular to its base, and *circular* because its base is a circle. But there are cones whose axes are not perpendicular to their bases, and cones whose bases are other curves than a circle.

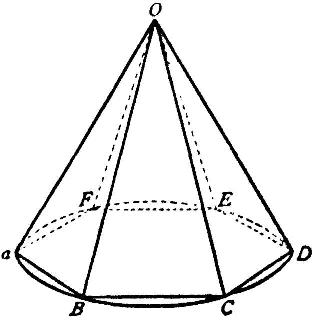
Extended Definition of a Cone.—A cone is the solid contained by a straight line which always passes through a fixed point, and some point on a fixed curve.

It is of course necessary that the fixed point should not be in the same plane as the curve.

101. *Connection between the pyramid and the cone.*

In the base of a cone inscribe a regular polygon, and join each of its angular points to the vertex O. Then we have a *symmetrical pyramid* inscribed in the cone.

By increasing the number of sides we can make the polygon which is the base of the pyramid differ by as little as we please from the base of the cone.



$\therefore$ by increasing the number of side-faces we can make the pyramid differ by as little as we please from the cone, both in surface and in volume.

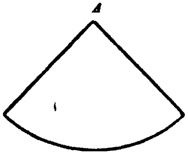
Therefore the cone may be considered the limit of a symmetrical pyramid, and all its properties deduced from it.

We may observe that

Cylinder : prism : : cone : pyramid.

102. *Surface of a cone.*

The surface of a cone consists of the base, which is a plane circle, and the curved surface, which is called the *lateral surface.*



Let ABC be the sector of a circle. If we cut it out of paper, and bend it round so that AB and AC meet, we shall practically have a cone. This will show that the lateral surface of a cone may be considered as equal to the sector of a circle, of which the radius AB represents the slant side, and the arc BC the circumference of the base.

But area of sector $= \frac{1}{2}$ arc $BC \times AB$. (Art. 39.)

$\therefore$ area of lateral surface of cone

$= \frac{1}{2}$ circumference of base × slant side.

$= \pi r l.$

If the whole exterior surface be required, we must add on the base, which is a circle with radius r.

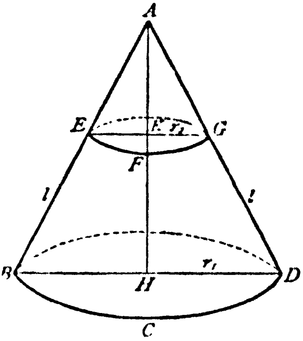
$\therefore$ total surface of cone $= \pi r l + \pi r^2.$

$= \pi r (l + r).$

103. *Frustum of a cone.*

Definition 19.—A frustum of a cone is that part cut off between the base and a plane parallel to it.

For instance, let the cone *ABCD* be cut by a plane *EFG* parallel to the base. The cone is divided into two parts, of which the upper part *AEFG* is a smaller cone, and the lower part *EFGBCD* a frustum of a cone.

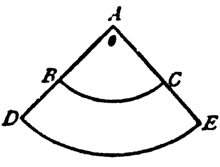


It is plain that the section of the plane with the cone will be a circle. The circles *EFG*, *BCD* are called the *ends of the frustum*; *KH*, the line joining their centres, the *altitude*; and any straight line *EB* the slant side.

If the plane *EFG* were not parallel to the base, we should obtain a more complicated figure, to which the name frustum of a cone might be given; but it is better to restrict the name, as in the case of the frustum of a pyramid. The student will observe that in the frusta of the pyramid and cone the cutting plane is parallel to the base, while in the frusta of the prism and cylinder it is not parallel.

104. *Lateral surface of frustum of cone.*

Let *ABC*, *ADE* be two concentric sectors. If we cut them out of paper, and bend them round so that *AD* and *AE* meet, we shall have a cone, of which *BDEC* will represent a frustum. Thus the surface of the frustum may be considered equal to *BDEC*, *BD* being the *slant side*, and *BC*, *DE* the circumference of the two ends.



Now area $BDEC =$ sector $ADE -$ sector ABC.

$$= \tfrac{1}{2}\{AD^2 - AB^2\}\,\theta = \tfrac{1}{2}(AD - AB)(AD\,.\,\theta + AB\,.\,\theta).$$

But $AD \times \theta = \text{arc } DE$, and $AB \times \theta = \text{arc } BC$. (Art. 36.)

$$\therefore \text{ area } BDEC = \frac{DB}{2} \times (\text{arc } DE + \text{arc } BC).$$

$\therefore$ lateral surface of frustum of cone

= ½ slant side × sum of circumference of bases.

If r_1 and r_2 are the radii of the two bases,

$$\text{Lateral surface} = \frac{l}{2}(2\pi r_1 + 2\pi r_2) = l\pi\,(r_1 + r_2).$$

Whole surface of frustum of cone

$$= l\pi\,(r_1 + r_2) + \pi r_1^2 + \pi r_2^2.$$
$$= \pi\,\{r_1\,(l + r_1) + r_2\,(l + r_2)\}.$$

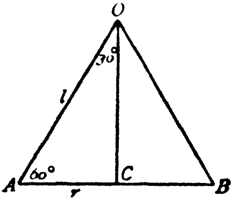
Examples.—(1) Find the whole surface of a cone whose altitude = 2 ft., and diameter of base = 7 ft. 6 in. ($\pi = 3{\cdot}1416$.)

We must first find the slant side (l).

$$l^2 = h^2 + r^2 = 24^2 + 45^2 \text{ sq. in.} = 2601 \text{ sq. in.}$$
$$\therefore\ l = 51 \text{ in.}$$
$$\therefore \text{ total surface} = \pi r\,(l + r) = \pi \times 45 \times (51 + 45).$$
$$= \pi\,.\,45 \times 96 \text{ sq. in.}$$
$$= 30\pi \text{ sq. ft.}$$
$$= 94{\cdot}248 \text{ sq. ft.}$$
$$= 94 \text{ sq. ft. } 36 \text{ in.} \quad \text{Answer.}$$

(2) Find the cost of the canvas for 500 conical tents, the vertical angle of each being 60°, and the radius of the base of each being 2 yds., at 5*d.* per sq. yd. ($\pi = 3{\cdot}1416$.)

Let the figure be a vertical section of one of the tents.



$\therefore$ angle $AOB = 60°$.

$\therefore$ angle $AOC = 30°$.

$\therefore$ $l = OA = 2AC = 4$ yds.

Lateral surface of one tent $= \pi rl$.

$= \pi \times 2 \times 4 = 8\pi$ sq. yds.

$\therefore$ cost of canvas for 1 tent $= 40\pi d$.

" for 500 tents $= 20{,}000\pi d$.

$= 62832d$.

= £261 16*s*. Answer.

(It is often convenient to draw a *section* of the figure only.)

(3) A steeple is 70 ft. high, and its slant side is 74 ft. Find approximately the cost of covering it with lead at 2*s.* 4*d.* per sq. ft.

We must first find the radius of the base (r).

$$r^2=l^2-h^2=74^2-70^2=4\times 144.$$

$$\therefore\ r=24 \text{ ft.}$$

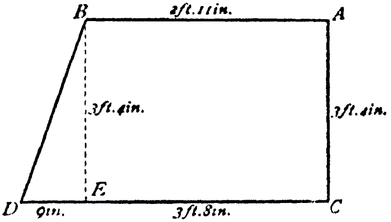
Surface of steeple $=\pi rl=\pi\,.\,24\times 74$ sq. ft.

$$\therefore\ \text{cost of lead}=\pi\,.\,24\times 74\times \tfrac{7}{3}s.$$

$$=\frac{22}{\cancel{7}}\times\overset{8}{\cancel{24}}\times 74\times\frac{\cancel{7}}{\cancel{3}}=176\times 74s.$$

$$=13024s.=£651\ 4s.\quad \text{Answer.}$$

(4) The two parallel sides of a trapezium are 2 ft. 11 in. and 3 ft. 8 in. The trapezium is made to revolve round its third side, 3 ft. 4 in. in length, which remains fixed. Find the lateral surface of the resulting solid.



The figure will be the frustum of a cone, the two parallel sides being the radii of the two ends, and the third side the altitude.

We must find the slant side.

$$BD^2=9^2+40^2=1681. \qquad \therefore\ BD=41 \text{ in.}$$

$$\text{Lateral surface}=l\pi\ (r_1+r_2).$$

$$=41\pi\ (35+44)=41\times 79\pi.$$

$$=3239\pi \text{ sq. in.}$$

If we take $\pi=\frac{22}{7}$, we obtain the result:

$$10180 \text{ sq. in.}=7 \text{ sq. ft. } 100 \text{ sq. in.}$$

This result is about 4 sq. in. greater than the exact result.

EXAMPLES ON THE CONE

(LENGTHS AND SURFACES)

1. The slant side of a cone is 4 ft. 10½ in., and the diameter of the base is 3 ft. 9 in. Find the altitude.

2. If the length of the axis of a cone is 4 yds., and the slant side is 4 yds. 6 in., find the area of the base.

3. If the vertical angle of a cone is 60°, and the slant side 20 ft., find the area of base. ($\pi = 3.1416$.)

4. If the vertical angle is $49°23'$, and the slant side 3 ft., find the altitude in ft. and decimal of a ft. ($\pi = 3.1416$, $\cos 24°41'30'' = .9086$.)

5. If the vertical angle is 45°, and the altitude 3 ft. 4 in., find the diameter of the base in ft. and decimal of a ft.

6. Find the lateral surface of a cone whose slant side is 3 ft. 10 in., and diameter of base 7 ft. 1 in.

7. Find the whole surface of a cone whose altitude is 5 ft. 3 in., and slant side 5 ft. 5 in.

8. Find the whole surface of a right-angled cone 5 ft. high. ($\pi = 3.1416$, $\sqrt{2} = 1.4142$.) Answer to four places of decimals.

9. The vertical angle of an acute-angled cone is 60°, and the diameter of the base is 17 ft. 8 in Find the lateral surface.

10. The lateral surface of a cone is 155 sq. ft. 120 in., and the radius of the base is 7 ft. Find the altitude.

11. A right-angled triangle, whose sides are 1 ft. and 5 in., revolves round the shorter side, which is fixed. Find the lateral surface of the resulting cone. ($\pi = 3.1416$.)

12. The diameters of the ends of the frustum of a cone are 60 ft. and 41 ft. 8 in., and the altitude is 50 ft. Find the slant side.

13. The diameters of the ends of the frustum of a cone are 13 ft. 5 in. and 4 ft. 1 in., and the slant side is 9 ft. 11 in. Find the altitude.

14. Find the lateral surface of the frustum of a cone, if the radii of the ends are 11 ft. and 4 ft., and the slant side is 15 ft.

15. Find the whole surface of the frustum of a cone, if the diameters of the ends are 25 yds. 1 ft. and 12 yds., and the altitude is 16 yds.

16. A trapezium, whose parallel sides are 8 in. and 3 in., revolves round its third side (perpendicular to first two), which is fixed and equal to 1 ft. Find in ft. and decimal of a ft. the whole surface of the solid of revolution thus formed. (π = [illegible]

17. The whole surface of a cone is 1256.64 sq. ft., and the slant side = 3 times the radius of base. Find the altitude to two places of decimals of a ft. ($\pi = 3.1416$.)

18. Find how long a pedestrian will take to ascend a conical hill whose height is 800 ft., and circumference of base 1 mile, at the rate of 10 miles in $5\frac{1}{2}$ hours.

19. What length of canvas, which is 1 yd. wide, will be required to make a conical tent 8 ft. in perpendicular height, with a radius of $6\frac{1}{2}$ ft. (*Sandhurst.*)

20. Find what length of canvas $\frac{3}{4}$ yds. wide is required to make a conical tent 7 yds. in diameter and 12 ft. high. (*Sandhurst.*)

21. Find the cost of the canvas required for an encampment for 6000 men, sleeping 6 in a tent, the tents being conical, 12 ft. high and 18 ft. in extreme breadth, canvas costing 5*d.* the sq. yd. ($\pi = 3.1416$.)

22. Find the cost of covering with lead a conical spire 90 ft. high, whose slant side is 90 ft. 6 in., at 13*s.* 6*d.* per sq. yd. ($\pi = 3.1416$.)

23. Find to the nearest cwt. the weight of the lead $\frac{1}{16}$ of an in. in thickness on a conical steeple 22 ft. in diameter of base, and 60 ft. high, if 1 cubic in. of lead weighs $6\frac{3}{5}$ oz.; and find also its cost at 20 guineas per ton.

24. A conical cap is formed by cutting a sector of a circle, whose arc is 13 in. and radius 10 in., out of paper, and bending the edges round so as to meet. Find the surface of the cap so formed.

25. A piece of paper in the form of a circular sector, of which the radius is 7 in. and the arc 11 in., is formed into a conical cap. Find the area of the conical surface, and also of the base of the cone. (*Sandhurst.*)

26. If the slope of a conical grassy hill 1200 ft. high takes 20 minutes to ascend, at the rate of $1\frac{3}{22}$ miles per hour, find the number of acres of grass on it to the nearest acre.

27. A conical hill, which stands on an area of $79\frac{6}{11}$ acres, has a flat top, the radius of which is 250 ft. Its ascent takes $6\frac{2}{3}$ minutes, at the rate of 3000 yds. an hour. Find the height of the hill.

28. A stone platform is in shape a conical frustum. Find to two places of decimals of a sq. ft. its total exterior surface, if its diameters at top and bottom are 5 ft. and 10 ft., and the slant side is 4 ft. 2 in. ($\pi = 3.1416$.)

29. The vertical angle of a cone is 140°28′, and the altitude is 27 in. Find the curved surface in sq. in. Given $\log 3 = .47712$, $\log \pi = .49715$, $L \sin 70°14' = 9.97362$, $L \cos 70°14' = 9.52916$, $\log 18844 = 4.27517$.

30. The lateral surface of a right-angled cone is 64 sq. ft. Find the altitude. Given $\log 2 = .30103$, $\log \pi = .49715$, $\log 37954 = 4.57925$.

Section II.—Volumes

[FORMULÆ:

(1) Volume of *cone* = $\frac{1}{3}$ base × altitude.

(2) Volume of *frustum of cone* = $\frac{\pi h}{3}\{r_1^2 + r_1 r_2 + r_2^2\}$

(where h = altitude, and r_1, r_2 the radii of the ends.)]

105. *Volume of cone* = $\frac{1}{3}\pi r^2 h$

(where h = altitude, r = radius of base).

Since the cone may be regarded as the limit of a symmetrical pyramid, when the number of sides is indefinitely increased, the volume of the pyramid will become ultimately equal to that of the cone.

$$\therefore \text{ volume of cone} = \tfrac{1}{3} \text{ base} \times \text{altitude}.$$
$$= \tfrac{1}{3}\pi r^2 h$$

(since the base is a circle whose radius is r).

Examples.—(1) Find the number of cubic yds. of earth in a conical barrow whose diameter at base is 60 ft., and whose slant side is 34 ft.

We must first find the altitude (h).

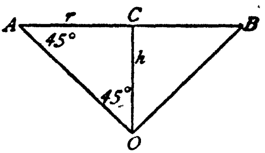
$$h^2 = l^2 - r^2 = 34^2 - 30^2 = 64 \times 4. \qquad \therefore h = 16 \text{ ft.}$$

Volume of cone $= \frac{1}{3} . \pi . 30^2 . 16$ cubic ft.

$$= \pi . \frac{300 \times 16}{27} = \frac{1600\pi}{9} \text{ cubic yds.}$$

$$= \frac{1600 \times 22}{9 \times 7} = 559 \text{ cubic yds. (to nearest yd.)} \quad \text{Answer.}$$

(2) Find correct to $\frac{1}{100}$ of an inch the depth of a conical wine-glass whose vertical angle is 90°, and which will hold $\frac{1}{4}$ pint.



Let the figure represent a vertical section of the wine-glass.

$\therefore$ angle $AOB = 90^\circ$.

$\therefore$ $AC = OC$.

i.e. $h = r$. h is the depth, which we have to find.

$$1 \text{ gallon contains } 277\tfrac{1}{4} = \tfrac{1109}{4} \text{ cubic in.}$$

$$\tfrac{1}{4} \text{ pint } \quad ,, \quad \frac{1109}{4 \times 32} = \frac{1109}{128} \text{ cubic in.}$$

$$\therefore \tfrac{1}{3}\pi \,.\, r^2h = \tfrac{1109}{128} \text{ cubic in.}$$

But $r = h$. $\quad \therefore h^3 = \dfrac{1109 \times 3}{128 \times \pi}$ cubic in.

$$h = \sqrt[3]{\frac{3327}{128\pi}} \text{ in.}$$

Since the answer is to be correct to three figures only, i.e. so many inches + two decimal places, we may take $\pi = \frac{22}{7}$.

If we do so, we shall obtain:

$$h = \sqrt[3]{\frac{23289}{2816}} = \sqrt[3]{8 \cdot 2702} \text{ \&c.} = 2 \cdot 02 \text{ in.} \quad \text{Answer.}$$

(Observe that the cone in this example was a right-angled cone. The student will easily see how to obtain the volume of a cone, if the vertical angles and *one* of the three lengths l, h, r are given.)

106. *Volume of frustum of cone* $= \dfrac{\pi h}{3}(r_1^2 + r_1r_2 + r_2^2)$.

Let h be the altitude, r_1, r_2 the radii of the two ends.

The frustum will be equal to the frustum of a symmetrical pyramid when the number of sides is indefinitely increased.

Now volume of frustum of pyramid $= \dfrac{h}{3}(B + \sqrt{BB'} + B')$, where B and B' are the two ends.

In the case of the cone, B and B' are circles whose radii are r_1 and r_2.

$\therefore$ $B = \pi r_1^2$, $B' = \pi r_2^2$, and consequently $\sqrt{BB'} = \pi r_1 r_2$.

$$\therefore \text{ frustrum of cone} = \frac{\pi h}{3}(r_1^2 + r_1r_2 + r_2^2).$$

Examples.—(1) A block of stone is in shape a conical frustum. The radii of the ends are 2 ft. and 3 ft., and the altitude is 4 ft. 6 in. Find the cost of the stone at 1*s.* 2*d.* per cwt., if every cubic ft. of stone weighs 168 lb.

Here $r_1=2$, $r_2=3$, $h=4\frac{1}{2}$.

$$\text{Volume of stone}=\frac{\pi\cdot 4\frac{1}{2}}{3}(4+6+9)\text{ cubic ft.}$$

$$=\pi\cdot\tfrac{3}{2}\cdot 19\text{ cubic ft.}$$

$$\text{Weight of stone}=\frac{3\times 19\pi}{2}\times 168\text{ lb.}$$

$$=\frac{3\times 19\pi\times 168}{2\times 112}\text{ cwt.}$$

$$\text{Cost of stone}\quad=\frac{14\times 3\times 19\pi\times 168}{2\times 112}d.$$

$$=\frac{\cancel{14}\times 3\times 19\times \overset{11}{\cancel{22}}\times \overset{3}{\cancel{168}}}{\cancel{2}\times \underset{2}{\cancel{112}}\times \cancel{7}}=1881d.$$

$$=£7\ \ 16s.\ 9d.\quad\text{Answer.}$$

(2) A mast is 30 in. in diameter at bottom, and 15 in. at top. If the mast contain $137\frac{1}{2}$ cubic ft. of wood, find its height in feet. ($\pi=\frac{22}{7}$.)

The mast will be in shape a conical frustum.

Let h = its height.

$$\text{Volume of mast}=\frac{\pi h}{3}\left\{\left(\tfrac{5}{8}\right)^2+\tfrac{5}{4}\cdot\tfrac{5}{8}+\left(\tfrac{5}{4}\right)^2\right\}.$$

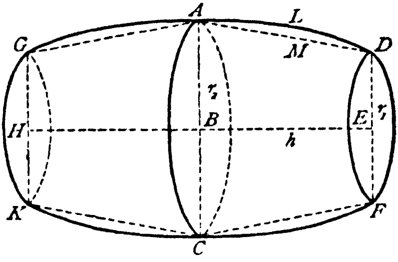
$$=\frac{22h}{21}\left(\tfrac{25}{64}+\tfrac{25}{32}+\tfrac{25}{16}\right).$$

$$=\frac{22h}{21}\times\frac{25\times 7}{64}=\frac{275h}{96}\text{ cubic ft.}$$

$$\therefore\ \frac{275h}{96}=137\tfrac{1}{2}=\tfrac{275}{2}.$$

$$\therefore\ h=\tfrac{96}{2}=48\text{ ft.}\quad\text{Answer.}$$

(3) There is another important application of the formula in question, though it is only an approximate one; i.e. its application to the gauging of casks.



Find the number of gallons in a cask whose diameter at either end is 18 in., whose diameter in the middle is 21 in., and which is 2 ft. long.

The cask may be approximately regarded as the sum of two equal frusta of a cone, joined together at the larger end.

Thus, in the figure, volume of cask approximately = twice the conical frustum whose altitude is BE, and the radii of whose ends are AB and DE.

Here $BE = 12$ in., $AB = 10\frac{1}{2}$ in., and $DE = 9$ in.

$$\therefore \text{ volume of cask} = 2 \cdot \frac{\pi \cdot 12}{3}\left\{9^2 + \tfrac{21}{2} \cdot 9 + (\tfrac{21}{2})^2\right\}.$$
$$= 8\pi\,(81 + \tfrac{189}{2} + \tfrac{441}{4}) \text{ cubic in.}$$
$$= 2\pi\,(324 + 378 + 441).$$
$$= 2\pi \times 1143 = 2286\pi \text{ cubic in.}$$

∴ number of gallons

$$= \tfrac{4}{1109} \times 2286\pi \text{ (for } 277\tfrac{1}{4} \text{ cubic in.} = 1 \text{ gallon).}$$

Taking $\pi = \frac{22}{7}$ and reducing, we find

number of gallons $= \frac{28738}{1109} = 26$ nearly. Answer.

(An examination of the figure will show that this volume is rather too small, for the slant side of the conical frustum passes straight from A to D, as AMD, while the side of the cask *bulges* out, as ALD. A more exact formula is given by writing r_2^2 for r_1r_2 in the formula for the frustum of a cone, so that

$$\text{volume of cask} = \frac{2\pi h}{3}\{r_1^2 + 2r_2^2\}.^*$$

If we apply this formula to the preceding example we shall obtain 27 gallons as the content of the cask. Hence we can be sure that the real content lies between 26 and 27 gallons.)

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[*] The advanced student will observe that this formula is obtained by considering the cask a frustum of a prolate spheroid.

EXAMPLES ON THE VOLUME OF THE CONE

1. Find the volume of a cone whose slant side is 4 ft. $10\frac{1}{2}$ in., and the radius of whose base is 1 ft. $10\frac{1}{2}$ in.

2. Find the volume of a cone whose altitude is 10 ft. 6 in., and whose slant side is 10 ft. 10 in.

3. Find the volume of a cone whose lateral surface is 3 sq. ft. 118 in., and slant side 2 ft. 1 in.

4. The vertical angle of a cone is 120°, and the slant side is 20 ft. Find the volume to a cubic ft. (π 3·1416.)

5. The volume of a cone is 440 cubic ft., and the altitude 8 yds. 1 ft. Find the diameter of the base.

6. The volume of a cone is 3 cubic yds. 18 ft., and the vertical angle is 30°. Find approximately the altitude, taking $\sqrt{3} = \frac{26}{15}$.

7. The volume of a cone is 333 cubic ft. 576 in., and the circumference of the base is 20 ft. Find the altitude.

8. The volume of a right-angled cone is 1050 cubic ft. Find the diameter of the base. (π 3·1416.)

9. The volume of a cone is $156\frac{4}{11}$ cubic ft., and radius of base = $3\frac{2}{7}$ times the height. Find the whole surface.

10. A right-angled triangle, whose hypotenuse is 1 ft. 5 in., and smaller side 8 in., revolves round its longer side. Find the surface and volume of the cone thus generated.

11. Find the volume of the double cone generated by the right-angled triangle, whose sides are 1 ft. 8 in. and 1 ft. 9 in., revolving round its hypotenuse.

12. Find the volume of the frustrum of a cone, the diameters of whose ends are 6 ft. and 4 ft. 2 in, and whose slant side is 5 ft. 1 in.

13. Find the volume of the frustum of a cone, if the lateral surface is $47\frac{1}{7}$ sq. in., and the radii of the ends are 4 in. and 2 in.

14. The volume of the frustum of a cone is 65 sq. yds. 5 ft., the altitude 5 ft., and the diameters of the ends are as 4 : 1. Find the lateral surface.

15. The radii of the ends of the frustum of a cone are 3 in. and 2 in., and the altitude is 5 in. Find the volume of the whole cone from which it is cut off. Prove your answer by showing that the volume of the frustrum = difference of the two cones.

16. A trapezium revolves round its side perpendicular to the other two. If this side is 9 ft., and the parallel sides are 5 ft. 10 in. and 4 ft. 2 in., find the volume of the solid generated correct to a cubic ft.

17. Find the number of cubic ft. of earth in a conical barrow, whose diameter at base is 60 ft., and slant side 30 ft. 6 in.

18. Find to the nearest ton the weight of a conical steeple 20 yds. high, and 6 yds. in diameter of base, if stone weighs 168 lb. to the cubic ft.

19. A conical tin vessel is made by cutting the sector of a circle from a thin sheet of tin, bending round the ends to meet, and then soldering them. If the arc of the sector is 22 in., and the radius of the circle is $12\frac{1}{2}$ in., find how much liquid the vessel will hold.

20. A conical wine-glass has a vertical angle of 45°, and is 2 in. in diameter at the top. Find to $\frac{1}{100}$ of an oz. what weight of water it will hold. ($\pi = 3.1416$.)

21. If the cost of covering a conical steeple, whose slant side is 37 ft., with lead at 1*s.* $5\frac{1}{2}$*d.* per sq. ft. be £101 15*s.*, find the cost of the stone if 1 cubic ft. of stone weighs 168 lb., and 1 ton of stone costs £1 3*s.* 4*d.*

22. Find the number of sq. ft. of canvas required for an encampment for 1200 men sleeping 6 in a tent, the tents being conical, 10 ft high, and requiring $513\frac{1}{3}$ cubic ft. of air.

23. A sugar-loaf in the shape of a cone, whose base diameter is 6 in., and slant side 18 in., costs 1*s.* 10*d.* Find the price per cubic in. in pence to three places of decimals.

24. The vertical angle of a conical wine-glass is 60°. Find to four decimal places of a pint the amount of water which must be poured into it to fill it to the depth of 2 in.

25. Find correct to the nearest lb. the weight of a conical frustrum of marble, whose top and bottom diameters are 1 ft. and 1 ft. 6 in., and height 1 ft., if 1 cubic ft. of marble weighs 2716 oz.

26. A tumbler is in shape an inverted frustum of a cone, the diameters of its two ends being $2\frac{1}{4}$ in. and 2 in. If the tumbler will hold $\frac{1}{2}$ pint of water, find its depth.

27. A tumbler is in shape an inverted frustum of a cone, the diameters of its two ends being $1\frac{3}{4}$ in. and $2\frac{1}{4}$ in., and its depth being $4\frac{1}{2}$ in. Find how many wine-glasses full it will hold if the wine-glass be conical, depth 2 in., and diameter at the top $2\frac{1}{2}$ in.

28. Find how many times a bucket, in shape an inverted conical frustum, should descend into a well in order to bring up 206 gallons. The depth of the bucket is 15 in., and the diameters of its ends are 12 in. and 10 in.

29. Find the height of a mast which is 48 in. in diameter at the bottom, and 40 in. in diameter at the top, and contains $476\frac{2}{3}$ cubic ft. of wood.

30. A temple is composed of three conical frusta set on each other. Their altitudes in order from bottom to top are 63 ft., 48 ft., and 24 ft., the radii of the top circles are in the same order, 60 ft., 30 ft., 10 ft., and the radii of the bottom circles in the same order are 76 ft., 44 ft., 17 ft. Find the whole exterior surface of the temple correct to a sq. ft., and its solid content correct to a cubic yd. ($\pi = 3.1416$.)

31. A tent is composed of the frustum of a cone surmounted by a smaller cone. The diameters of the ends of the frustrum are 8 ft. and 4 ft.; the total height of the tent is 13 ft. 6 in.; the height of frustum 10 ft. Find the number of cubic ft. of air inside the tent.

32. Find approximately the number of gallons in a beer-cask, whose diameter at end is 14 in., circumference of greatest section $49\frac{1}{2}$ in., and length 15 in.

33. Find approximately the number of gallons in a cask, if the end circumference is 66 in., the circumference of greatest section 77 in., and the length 2 ft.

34. Find approximately the number of gallons in a cask, whose end circumference is 3 ft. $2\frac{1}{2}$ in., circumference of greatest section 4 ft. 7 in., and the length 1 ft. 9 in.

35. If the radii of the ends of a cask are each 13 in., and the radius at the centre is 16 in., the length being 42 in., show that the number of gallons in the cask lies between 100 and 108.

36. Find to four decimal places of a cubic ft. the volume of a right-angled cone, whose whole surface is 729 sq. ft. Given $\log 3 = .4771213$, $\log \pi = .4971500$, $\log (\sqrt{2} + 1) = .3827756$, $\log 98680 = 4.9942291$, $D = 44$.

37. The vertical angle of a cone is $58°30'$. If the altitude is 10 in., find the volume. Given $\log 3 = .4771213$, $\log \pi = .4971500$, $L \tan 29°15' = 9.7482089$, $\log 3.2843 = .5164428$, $D = 132$.

38. Find the height in ft. of a conical steeple, whose vertical angle is $12°$, if the weight of the stone composing it be 400 tons at 166 lb. per cubic ft. Given $\log 2 = .30103$, $\log 3 = .47712$, $\log 7 = .84510$, $\log \pi = .49715$, $\log 83 = 1.91908$, $L \tan 6° = 9.02162$, $\log 77.561 = 1.88964$.

39. The vertical angle of a conical tent is $68°$. The canvas of which it is made costs 6*s.* 8*d.*, at $4\frac{1}{2}$*d.* per sq. yd. Find the number of cubic ft. of air in the tent. Given $\log 2 = .30103$, $\log 3 = .47712$, $\log \pi = .49715$, $L \cot 34° = 10.17101$, $L \operatorname{cosec} 34° = 10.25244$, $\log 23596 = 4.37284$.

40. The vertical angle of a tent containing 300 cubic ft. of air is $50°$. Find the number of sq. ft. of canvas in the tent to three decimal places. Given $\log 3 = .4771213$, $\log \pi = .4971500$, $L \cot 25° = 10.3313275$, $L \operatorname{cosec} 25° = 10.3740517$, $\log 1942.5 = 3.2883610$, $D = 224$.

CHAPTER VII.—THE SPHERE

Section I.

[FORMULÆ :

(1) Surface of *sphere* (radius $= r$) $= 4\pi r^2$.

(2) Surface of *zone or segment of sphere* $= 2\pi rh$ (where $h =$ altitude).]

107. DEFINITION 20.—"A sphere is a solid figure described by the revolution of a semicircle about its diameter, which remains fixed."

"The *axis* of a sphere is the fixed straight line about which the semicircle revolves."

"The *centre* of a sphere is the same with that of the semicircle."

"The *diameter* of a sphere is any straight line which passes through the centre, and is terminated both ways by the superficies of the sphere." (Euclid xi. def. 14–17.)

The *axis* of a sphere is a term not much in use. The *radius* of a sphere is the straight line drawn from the centre to the surface.

It is unnecessary to give any illustration of so well known a figure as a sphere. A globe or a round ball are familiar instances.

108. A sphere may also be defined as a solid, every point in whose surface is equidistant from a fixed point called the centre, or as the locus of a point which moves so as to always keep the same distance from a fixed point.

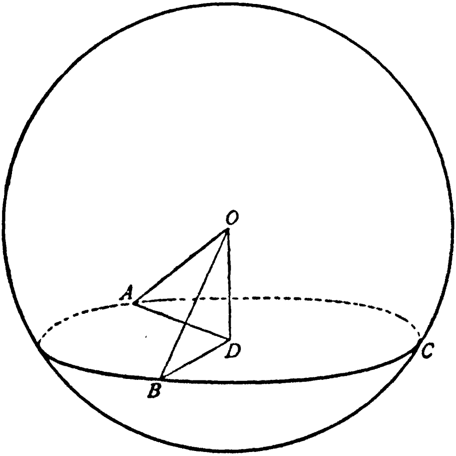
109. *Surface of sphere* $= 4\pi r^2$.

This formula may be simply expressed by the rule: *Multiply the diameter by the circumference.*

The student should notice that the surface $= \frac{2}{3}$ *of the circumscribing cylinder*, the base of which would be πr^2, and the height $2r$; so that its total surface would be $4\pi r^2 + 2\pi r^2 = 6\pi r^2$. The proof of this formula does not admit of any simple illustration. (See Art. 112.)

110. *Every section of a sphere by a plane is a circle.*

Let ABC be the section of the sphere, whose centre is O, by any plane. Draw OD at right angles to the plane.



Take A and B, *any* two points on the section, and join OA, AD, OB, BD.

Then the angles ODA, ODB are right angles, and OA, OB are radii of the sphere.

$$\therefore AD^2 = OA^2 - OD^2 = OB^2 - OD^2 = BD^2.$$

$$\therefore AD = BD.$$

Now *A*, *B* are *any* points on the section; so that we can similarly prove that every point on the section is equidistant from *D*. Thus the section is a circle, of which *D* is the centre. Q.E.D.

If the plane passes through the centre of the sphere, its section is called a *great* circle. A great circle divides the sphere into two *hemispheres*.

111. *Zone and segment of sphere.*

DEFINITION 21.—If a sphere is cut into two parts by a plane, each part is called a *segment.*

A *zone* of a sphere is the part contained between two parallel planes.

In a segment the circular section made by the cutting-plane is called the *base* of the segment; in a zone the sections made by the cutting-planes are called the *bases.* A segment may be regarded as a zone, one of whose bases has zero for radius.

The *altitude of a segment* of a sphere is the straight line drawn from the centre of the base, at right angles to it, to the surface of the sphere.

The *altitude of a zone* of a sphere is the straight line joining the centres of the bases.

In either case the altitude is an intercepted part of a diameter of the sphere.

112. *Curved surface of zone or segment* $= 2\pi rh$ (where $h =$ the altitude.)

Since the segment is really a particular case of the zone, the formula is the same for both.

This formula also does not admit of any simple illustration. It offers the remarkable result that *the curved surface of a zone or segment depends on the altitude alone,* irrespective of the size of the bases.

If we wish to express the whole surface we must add on the area of the base or bases.

If we take the diameter for the altitude, we obtain surface of sphere $= 2\pi r \times 2r = 4\pi r^2$ as before.

Examples.—(1) Find the expense of gilding a ball 5 ft. in diameter, at 6*s.* 3*d.* per sq. ft. $\pi = 3 \cdot 1416$. (*Sandhurst.*)

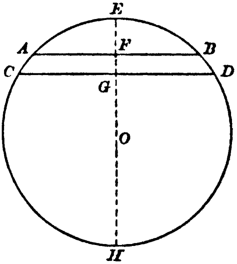
Here $r = \frac{5}{2}$ ft.

Surface $= 4\pi(\frac{5}{2})^2 = 25\pi = 78 \cdot 54$ sq. ft.

$\therefore$ cost $= 78 \cdot 54 \times 6\frac{1}{4}s. = 490 \cdot 875s. = £24$ 10*s.* 10½*d.* Answer.

(2) Find the areas of the three portions into which the surface of a sphere, whose diameter is 2 ft., is divided by two parallel planes which cut the diameter at right angles to them in the proportion of 9 : 1 : 2.

Let the figure represent a section of the sphere at right angles to the cutting-planes.



Then the diameter is divided into the portions $EF = \frac{1}{6} EH = 4$ in., $FG = \frac{1}{12} EH = 2$ in., and $GH = \frac{3}{4} EH = 18$ in.

$\therefore$ surface of segment AEB
$= 2\pi \,.\, 12 \times 4 = 96\pi$ sq. in.

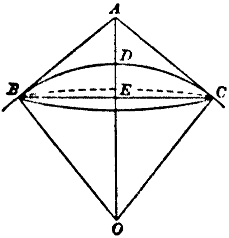
surface of zone $ACDB$
$= 2\pi \,.\, 12 \times 2 = 48\pi$ sq. in.

surface of segment CHD
$= 2\pi \,.\, 12 \times 18 = 432\pi$ sq. in.

These three areas, correct to a sq. in., become:

2 sq. ft. 14 in.; 1 sq. ft. 7 in.; 9 sq. ft. 61 in. Answer.

(3) Taking the earth's radius at 4000 miles, find how many yards above the surface of the earth a balloon must be in order that there may be visible from it an area of 11,000 sq. miles.



Let A be the balloon, O the centre of the earth.

Then area of zone $BDC = 11{,}000$ sq. miles; that is,

$$2\pi r \times DE = 11{,}000.$$

$$\frac{\quad}{2\pi \times 4000} \quad \frac{\quad}{8\pi}$$; miles.

We have to find AD, the height of balloon.

$$= 4000 - \frac{7}{18} \text{ miles.}$$

But $OA : OB :: OB : OE$ (by similar triangles).

$$\therefore\ OA = \frac{4000^2}{4000 - \frac{7}{16}}.$$

$$AD = OA - OD = \frac{4000^2}{4000 - \frac{7}{16}} - 4000 \text{ miles.}$$

$$= \frac{\frac{7}{16} \times 4000}{4000 - \frac{7}{16}}$$

$$= \frac{28000}{63993} \text{ miles.}$$

$$= \frac{28000 \times 1760}{63993} \text{ yds.}$$

If this result is worked out to the nearest yd., we find:

Height of balloon = 770 yds. Answer.

EXAMPLES ON THE SPHERE (SURFACE, ETC.)

(a) Surface of Sphere

1. Find the surface of sphere whose radius is 10 in. $(\pi = 3.1416.)$

2. Find the surface of sphere, the circumference of a great circle of which is 3 ft. 8 in.

3. The surface of a sphere is 68 sq. ft. 64 in. Find the diameter.

4. The surface of a sphere is 17 sq. ft. 16 in. Find the circumference of any great circle.

5. Every point on a sphere is 2 ft. distant from a point inside it. Find the surface correct to three decimal places of a ft. $(\pi = 3.1416.)$

6. A semicircle, whose diameter is 100 yds., revolves round its diameter. Find the surface of the sphere generated in acres and sq. yds. $(\pi = 3.1416.)$

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7. Find the cost of gilding a ball 7 ft. in diameter, at 7*s.* the sq. ft.

8. The cost of gilding a ball 21 ft. in diameter is £519 15*s.* What is that per sq. ft.?

9. Find the cost of the material for making a spherical balloon 14 yds. 2 ft. in circumference, at 9*s.* per sq. yd.

10. Find approximately the exterior surface of the earth, taking the circumference of a great circle as 24,900 miles. $(\pi = 3.1415927.)$

11. Find the amount of material required to cover a spherical football with leather, if the circumference of the football is 2 ft. $(\pi = 3.1416.)$

12. Sixteen equal cannon-balls, placed side by side in a straight line, extend to a distance of 6 ft. Find the surface of one of the shot. $(\pi = 3.1416.)$

(b) Zone and Segment of Sphere

13. The altitude of the segment of a sphere is 3 ft. If the radius of the sphere is 5 ft., find the curved surface of segment. ($\pi = 3.1416$.)

14. If the surface of a sphere is 88,704 sq. ft, and the curved surface of the zone of the sphere is 15,840 sq. ft., find the altitude of zone.

15. Find the altitude of a segment of a sphere whose radius is 25 ft., if the curved surface of the segment is $1570\frac{4}{5}$ sq. ft. ($\pi = 3.1416$.)

16. If the curved surface of a zone of a sphere is 33 sq. in., and its altitude is $1\frac{1}{2}$ in., find the surface of the whole sphere.

17. Find the area of the base of a segment of a sphere, if the altitude of the segment is 1 in., and the radius of the sphere 2 ft. 1 in.

18. The radii of the two bases of a zone are 15 ft. and 7 ft., and the radius of the sphere is 25 ft. Find the whole surface of the zone. Show that there are two solutions, and explain the reason.

19. Given radius of sphere = 5 ft. 5 in., altitude of zone = 3 in., and radius of the larger base - 2 ft. 1 in., find the whole surface of the zone.

20. Find the whole surface of the segment of a sphere whose altitude is 4 in., if the radius of the sphere is 2 ft. 10 in.

21. The radius of a sphere is 15 in. Find the whole surface of a zone which is the difference between two segments having their altitudes $\frac{2}{3}$ in. and 3 in. respectively.

·22. The surface of a sphere is 616 sq. yds. A plane cuts it at a distance of 4 yds. from the centre. Find the area of the section made by the cutting-plane.

23. Find the altitude of a zone of a sphere whose radius is 24 ft., if its surface is equal to the whole surface of a sphere whose radius is 8 ft.

24. Find to the nearest sq. ft. the exterior surface of a dome, in shape a segment of a sphere greater than a semicircle, if the height is 27 ft., and the diameter of the base 12 ft.

25. Taking the earth's radius as 4000 miles, find what fraction of the earth's surface is visible from a balloon at the height of 1000 ft.

26. Taking the earth's radius as 4000 miles, find how many feet from the surface of the earth a balloon must be in order that there may be in view from it an area of 5000 sq. miles.

27. Find how far distant from the surface of a sphere a person must be to see $\frac{3}{8}$ of it, if the diameter of the sphere is 16 ft.

28. Find how far distant a person must be from the centre of a spherical body, whose diameter is 10 ft., in order to see $\frac{1}{8}$ of it.

29. The moon is 240,000 miles distant from us, and has a diameter of 2160 miles. What portion of the moon's surface is visible to us?

30. Find the area of the 10 degrees in latitude nearest to the North Pole, if the diameter of the earth be taken as 7900 miles, and the earth be regarded a perfect sphere. Given $\log 79 = 1.8976271$, $\log \pi = .4971500$, $L \sin 5^\circ = 8.9402960$, $\log 14893 = 4.1729822$, $\log 14894 = 4.1730113$.

Section II.—Volumes

[FORMULÆ:

(1) Volume of *sphere* (radius r) $= \frac{4}{3}\pi r^3$.

(2) Volume of *zone of sphere* $= \frac{\pi h}{6}\{3(r_1^2 + r_2^2) + h^2\}$ (where h = altitude, r_1, r_2 = radii of the bases).

(3) Volume of *segment of sphere* $= \frac{\pi h}{6}\{3r_1^2 + h^2\}$.]

113. *Volume of sphere* $= \frac{4}{3}\pi r^3$.

Assuming that the surface of sphere $= 4\pi r^2$, we can deduce the volume.

Suppose a large number of points taken on the surface of a sphere, and that a tangent plane is drawn through each of these points. Then these planes will form by their intersections a series of plane figures, which will be the faces of a polyhedron described about the sphere. Join each angular point to the centre of the sphere; then volume of polyhedron = sum of the volumes of a series of pyramids, which have a common vertex at the centre of the sphere, and whose bases are the faces of the polyhedron. But volume of each pyramid $= \frac{1}{3}$ altitude × face of polyhedron, and the common altitude of all the pyramids is the radius r.

$$\therefore \text{ volume of polyhedron} = \frac{r}{3} \times \text{ sum of the faces.}$$

$$= \frac{r}{3} \times \text{surface of polyhedron.}$$

Now, by indefinitely increasing the number of faces, the surface and volume of the polyhedron can be made as nearly as we please equal to the surface and volume of the sphere.

$$\therefore \text{ volume of sphere} = \tfrac{1}{3} \text{ radius} \times \text{surface.}$$

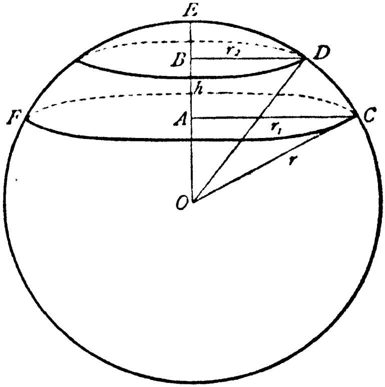
$$r \qquad {}^{2} = 4\pi r^{3}. \qquad \text{Q.E.D.}$$

114. *Volume of zone of sphere.*

If r_1, r_2 the radii of the bases, and h the altitude are given,

$$\text{Volume} = \frac{\pi h}{6}\{3(r_1^2 + r_2^2) + h^2\}.$$

This formula admits of no simple illustration.



If the radius of the sphere is known, the volume can be expressed in terms of the three radii; or of the altitude, the radius of the sphere, and the radius of either base.

Let A, B be the centres of the two bases of a zone, both lying on the same side of O the centre of the sphere.

Then $h = AB = OB - OA = \sqrt{OD^2 - BD^2} - \sqrt{OC^2 - CA^2}$.

$$\therefore\ h = \sqrt{r^2 - r_1^2} - \sqrt{r^2 - r_2^2}.$$

If the cutting-planes are on opposite sides of O, then

$$h = \sqrt{r^2 - r_1^2} + \sqrt{r^2 - r_2^2}.$$

Thus the altitude, and consequently the volume, may be expressed in terms of the three radii.

Similarly r_2 can be expressed in terms of r, r_1, and h.

115. *Volume of segment of sphere.*

If r_1 is the radius of the base, and h the altitude, we have only to put $r_2 = 0$ in the former formula.

$$\therefore\ \text{volume} = \frac{\pi h}{6}\{3r_1^2 + h^2\}.$$

If the radius of the sphere is known, the volume can be expressed in terms of the altitude and radius.

Suppose that in the above figure we have to find the volume of the smaller segment cut off by the plane FAC. Then $h = AE$.

$$r_1^2 = OC^2 - OA^2 = r^2 - (r-h)^2 = 2hr - h^2.$$

Substituting this value for r_1^2 in the formula, we obtain

$$\text{Volume of segment} = \frac{\pi h}{6}\{6hr - 3h^2 + h^2\}.$$

$$= \frac{\pi h^2}{6}\{6r - 2h\}.$$

$$= \frac{\pi h^2}{3}(3r - h).$$

Similarly the volume may be expressed in terms of r and r_1, but the expression is more complicated.

N.B.—The volume of a sphere = $\frac{2}{3}$ of that of the circumscribing cylinder.

Examples.—The practical applications of the sphere are mainly concerned with finding the weight of shot and shell.

(1) If one cubic in. of iron weigh 4·2 oz., find the weight of an iron shot whose diameter is 7 in. Also find the radius of a 20 lb. shot.

$$\text{Volume of shot} = \tfrac{4}{3}\pi \cdot (\tfrac{7}{2})^3 = \frac{\cancel{4}}{3} \cdot \frac{343}{\underset{2}{\cancel{8}}}\pi = \frac{343\pi}{6} \text{ cubic in.}$$

$$\therefore \text{ weight of shot} = \frac{343\pi}{6} \times 4{\cdot}2 \text{ oz.}$$

$$= \frac{343}{\cancel{6}} \times \frac{\overset{11}{\cancel{22}}}{\cancel{7}} \times \frac{\cancel{42}}{\underset{80}{\cancel{160}}} \text{ lb.}$$

$$= \frac{3773}{80} \text{ lb.}$$

$= 47$ lb. (rather more). **Answer.**

Again, weight of shot $= 20 \times 16$ oz.

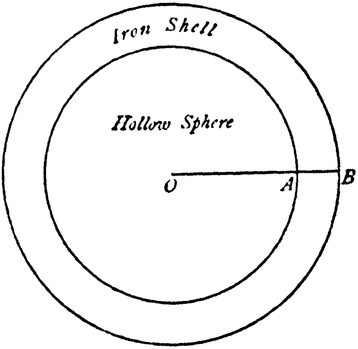
$$\therefore \text{ volume of shot} = \frac{20 \times 16}{4\cdot 2} \text{ cub. in. } . = \frac{20 \times 160}{42}.$$

$$\tfrac{4}{3}\pi r^3 = \frac{20 \times 160}{42}. \qquad \therefore\ r^3 = \frac{20 \times 160 \times 3}{4 \times 42 \times \pi} = \frac{400}{7\pi}.$$

$$r = \sqrt[3]{\tfrac{400}{22}} = \sqrt[3]{18\tfrac{2}{11}} = 2\cdot 63 \text{ in., \&c.} \quad \text{Answer.}$$

(2) Find the weight of a hollow iron shell, if the exterior diameter is 13 in., and the thickness of the iron 2 in. Iron weighs 4·2 oz. per cubic in.

The volume of a hollow shell is always the difference of two spheres, one having the outer diameter of the shell, and the other the inner diameter.



Thus in the figure the volume of the shell is the difference of the volumes of two spheres whose radii are OB and OA. AB, the difference of their radii, is called the thickness of the shell.

In the present example $OB = \frac{13}{2}$, $AB = 2$.

$\therefore\ OA = \frac{9}{2}$.

$$\text{Volume of shell} = \frac{4\pi}{3}\left\{\left(\tfrac{13}{2}\right)^3 - \left(\tfrac{9}{2}\right)^3\right\} \text{ cubic in.}$$

$$= \frac{4\pi}{3} \cdot \frac{1468}{8} \text{ cubic in.}$$

$$\therefore \text{ weight of shell} = \frac{\cancel{4}}{\cancel{3}} \cdot \frac{\overset{11}{\cancel{22}}}{\cancel{7}} \cdot \frac{\overset{367}{\cancel{1468}}}{\cancel{8}} \cdot \frac{\overset{\cdot 2}{\cancel{4\cdot 2}}}{\cancel{16}} \text{ lb.}$$

$$20$$

$$= \frac{367 \times 11}{20} = 202 \text{ lb. (to nearest lb.)} \quad \text{Answer.}$$

(3) A spherical shell of iron, whose diameter is 1 ft., is filled with lead. Find the thickness of the iron, when the weights of iron and lead are equal. A cubic inch of iron weighs 4·2 oz., and a cubic inch of lead weighs 6·6 oz. (*Sandhurst.*)

See figure in last example. Here $OB = 6$ in.

Let x = thickness of iron (AB).

$\therefore$ volume of lead $= \frac{4}{3}\pi\,(6-x)^3$ cubic in.

„ iron $= \frac{4}{3}\pi\,\{6^3-(6-x)^3\}$ cubic in.

$\therefore$ weight of lead $= \frac{4}{3}\pi\,(6-x)^3 \,.\, 6\frac{3}{5}$ oz.

„ iron $= \frac{4\pi}{3}\,\{6^3-(6-x)^3\}\;4\frac{1}{5}$ oz.

$$\therefore \frac{4\pi}{3}(6-x)^3 \,.\, \frac{33}{5} = \frac{4\pi}{3}\{6^3-(6-x)^3\}\,\frac{21}{5}.$$

$$(6-x)^3 \,.\, 11 = \{6^3-(6-x)^3\}\,7.$$

$$18\,(6-x)^3 = 7 \,.\, 6^3.$$

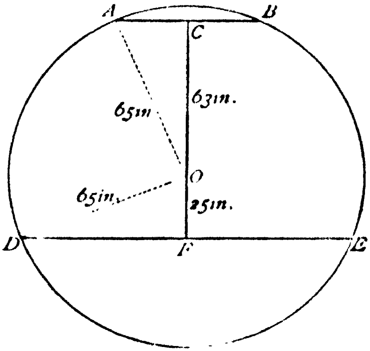
$$6-x = \sqrt[3]{\tfrac{7}{18}} \,.\, 6.$$

$$x = 6\,(1-\sqrt[3]{\tfrac{7}{18}}).$$

Extracting the cube root of $\frac{7}{18}$ to two places of decimals, we have: Thickness of iron $= 6\,(1 - \cdot 73)$ in.

$= 1{\cdot}62$ in. Answer.

(4) The radius of a sphere is 5 ft. 5 in. Find the volume contained between two parallel planes which cut it on different sides of the centre, and at distances of 5 ft. 3 in. and 2 ft. 1 in. from it respectively.



The volume is a zone whose altitude

$= 5$ ft. 3 in. $+ 2$ ft. 1 in.

$= 7$ ft. 4 in.

$= 88$ in.

The radii of the bases are AC and DF in the figure.

$$AC = \sqrt{65^2-63^2} = \sqrt{2\times128} = 16 \text{ in.}$$

$$DF = \sqrt{65^2-25^2} = \sqrt{90\times40} = 60 \text{ in.}$$

$$\therefore \text{volume} = \frac{\pi\,.\,88}{6}\{3\,(16^2+60^2)+88^2\} \text{ cubic in.}$$

$$= \frac{\pi\,.\,88\times19312}{6} \text{ cubic in.}$$

If we take $\pi = \frac{22}{7}$, and work out the answer to the nearest cubic ft., we obtain: Volume of zone $= 515$ cubic ft. Answer.

EXAMPLES ON THE VOLUME OF THE SPHERE

(a) Volume of Sphere

1. Find the volume of a sphere whose radius is 9 in.; and if the volume is 1000 cubic ft., find the radius.

2. Find the volume, if the circumference of a great circle is 7 ft. 4 in.

3. If the volume is 11,498$\frac{2}{3}$ cubic yds., find the circumference of a great circle.

4. If the volume is 53$\frac{19}{81}$ cubic in., find the surface; and if the surface is $\frac{1}{10}$ of an acre, find the volume.

5. A semicircle, whose diameter is 10 ft., revolves round its diameter. Find the volume of the resulting sphere. ($\pi = 3.1416$.)

6. Determine to the nearest hundredth of an inch the radius (1) of a sphere whose volume is 1 cubic ft., (2) of a sphere whose surface is 1 sq. ft. ($\pi = 3.1416$.) (*Sandhurst.*)

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7. A sphere, whose diameter is 1 ft., is cut out of a cubic ft. of lead, and the remainder is melted down into the form of another sphere. Find its diameter. ($\pi = 3.1416$.) (*Sandhurst.*)

8. Find the weight of an iron cannon-ball 5 in. in diameter, if 1 cubic in. of iron weighs 4.2 oz.

9. Determine the number of yds. of material necessary to make a spherical balloon containing 1000 cubic ft. of gas. (*Sandhurst.*)

10. The number of sq. yds. of silk required for making a spherical balloon is 55.44. Find the number of cubic ft. of gas it contains.

11. Find the value of a ball of pure gold 9 in. in diameter, if 1 cubic ft. of gold weighs 19,300 oz. avoirdupois, and

1 oz. Troy of gold is worth £3 18*s*. ($\pi = 3.1416$.) Answer to the nearest £.

12. A cubic ft. of gold weighs 19,300 oz. avoirdupois, a cubic ft. of copper 8890 oz. Find the exterior surfaces of two spheres, one of gold, the other of copper, each weighing 14 lb., to three decimal places of a sq. ft. Which of the two has the larger surface, and by how many sq. in.?

13. A hemispherical basin, whose diameter is 18 in., is filled to the depth of 6 in. with water. Find the weight of the water in the basin.

14. What is the weight of a hollow sphere of metal whose inside diameter is 1 ft., and thickness 1½ in.? Given that a cubic ft. of the metal weighs 7776 oz.

15. What is the weight of a hollow sphere of metal whose inside diameter is 1½ ft., and thickness 2 in.? Given that a cubic ft. of the metal weighs 7776 oz. (*Sandhurst.*)

16. If 30 cubic in. of gunpowder weigh 1 lb., find the weight of gunpowder required to charge a shell, the exterior diameter of which is 10 in., and the thickness of the iron 1 in.

17. Find the difference in weight between a shell of iron and a shell of lead of the same size, each having a diameter of 1 ft., and thickness of 2 in. A cubic in. of iron weighs 4.2 oz., and an oz. of lead contains ·15 of a cubic in.

18. The thickness of the iron in a shell is 2 in. If the volume = $\frac{2}{7}$ of a cannon-ball whose diameter is 14 in., find the external diameter of the shell.

19. Compare the weights of a solid cannon-ball and a shell, the external diameter of both being 6 in., but the internal diameter of the shell being 4½ in.

20. A solid cannon-ball, whose diameter is 4 in., is 3 times the weight of a shell. If the thickness of the iron in the shell is 1 in., find its external diameter.

21. Find the solid content of a spherical shell, the inner and outer diameters of which are 6 in. and 4 in.; also its weight, if a cubic ft. weighs 7776 oz. ($\pi = 3.1416$.)

22. Find the weight of a shell, the metal being 1 in. thick, and weighing 486 lb. to the cubic ft., if the exterior diameter is 5 in.

23. A ball of iron 4 in. in diameter weighs 9 lb., and a ball of lead 1 in. in diameter weighs $\frac{3}{14}$ lb. Find the weight of a ball composed of an iron sphere 8 in. in diameter, coated with a layer of lead 7 in. thick. (*Sandhurst.*)

24. Assuming that a cubic ft. of water weighs 1000 oz., and that a given volume of iron weighs 7.21 times as much as the same volume of water, find the weight of a bombshell, the exterior and interior diameters being 10 in. and 8 in. respectively. (*Sandhurst.*)

25. The exterior diameter of a shell is 1 ft., the interior diameter is 9 in. Find its weight when filled with gunpowder, if $4\frac{1}{2}$ oz. of iron and $\frac{1}{2}$ oz. of gunpowder go severally to the cubic in.

26. Determine the thickness of the iron in a shell whose exterior diameter is 6 in., if when filled with gunpowder it weighs $\frac{1}{6}$ more than when empty, iron and gunpowder weighing as in the last example. Answer to three decimal places of an inch.

27. A ball of lead 4 in. in diameter is covered with gold. Find the thickness of the gold in order that (1) the volumes of gold and lead may be equal, (2) the surface of the gold may be twice that of the lead. (*Sandhurst.*)

(b) Volumes of Zone and Segment

28. Find the volume of the zone of a sphere, if the diameters of the bases are 3 ft. and 5 ft., and the altitude 9 in. ($\pi = 3.1416$.)

29. Find the volume of the segment of a sphere, the diameter of whose base is 2 ft. 6 in., and altitude 15 in. ($\pi = 3.1416$.)

30. The altitude of the segment of a sphere is 6 in., and the radius of the sphere is 9 in. Find the volume of the segment.

31. The diameter of a sphere is 50 ft., and the diameters of the bases of a zone, which does not contain the centre of the sphere, are 40 ft. and 14 ft. Find the volume of the zone to the nearest cubic ft. ($\pi = 3.1416$.)

32. The radii of the base of a zone of a sphere are 4 in. and 1.4 in., and the altitude is 1.8 in. Find the volume of the sphere of which it is a section. ($\pi = 3.1416$.)

33. The volume of a segment of a sphere is 198 cubic in., and the altitude is 3 in. Find the volume of the whole sphere approximately.

34. The longer radius of the bases of a zone is 7 ft., and the radius of the sphere is 7 ft. 1 in. Find the volume of the zone, if its altitude is 6 ft. 9 in. Is this zone greater or less than a hemisphere?

35. A plane cuts a diameter of a sphere, 8 in. in length, at right angles, dividing it in the ratio of 3 : 5. Find the volume of each part.

36. Find the volume of the zone of a sphere whose radius is 17 in., which is the difference of two segments, which are both on the same side of the centre, and 8 in. and 16.8 in. distant respectively from the centre.

37. The diameter of the base of the segment of a sphere, smaller than a hemisphere, is 3 ft. 6 in., and the diameter of the sphere is 4 ft. 10 in. Find the volume of the segment.

38. A champagne-glass is in shape a segment of a sphere. The diameter of the top is 3⅛ in., and the depth is 2 in. Find the content of the glass to five decimal places of a pint.

39. A Dutch cheese is spherical, the radius being 5 in. If 7 lb. of cheese contain 419 cubic in., determine the weight of cheese in a zone, the radii of whose bases are 3 in. and 1⅔ in.

40. If the radius of a Dutch cheese be 5 in., and the radii of a zone are 4 in. and 1⅔ in., and 5 lb. of cheese contain 306 cubic in., find the weight of the zone.

CHAPTER VIII.—SIMILAR SOLIDS

Section I.

116. *Similar solids* are those which have the same *shape*, but not necessarily the same size.

In treating of similar solids it will be convenient to remember the distinction drawn in the first chapter between the two kinds of solids we have considered; i.e. *polyhedra* and *solids of revolution*.

117. *Similar polyhedra.*

DEFINITION 22.—"Similar solid figures are such as have all their solid angles equal, each to each, and are contained by the same number of similar planes." (Euclid xi. def. 11.)

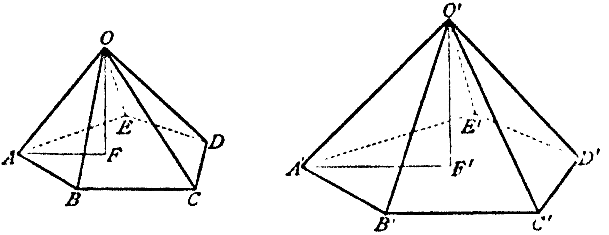
This definition is not an easy one. It really applies only to polyhedra; i.e. to solid figures whose faces are plane figures. In fact Euclid supplements it with another definition for cones and cylinders, which we shall give presently.

The definition will be made plainer by substituting *plane figures* for planes. The amended definition may run:

"Two polyhedra are similar when their solid angles are equal, each to each, and they are contained by the same number of similar plane figures."

All regular polyhedra of the same number of sides are similar. Thus all cubes are similar figures; also all tetrahedra and octahedra.

118. Let $OABCDE$, $O'A'B'C'D'E'$ be two similar polyhedra; then the solid angles at O, A, B, C, D, E are equal to the solid angles at O', A', B', C', D', E', each to each, and every face of one polyhedra is similar to the corresponding face of the other.



Now, since the faces are similar, it follows that *the edges of one polyhedron are proportional to the corresponding edges of the other.*

For instance, $AO : CD :: A'O' : C'D'$.

For from similar figures, $AO : AB :: A'O' : A'B'$,

and $AB : CD :: A'B' : C'D'$;

$\therefore AO : CD :: A'O' : C'D'$.

Also, since the areas of similar figures are proportional to the squares of their sides, it follows that the corresponding faces of the polyhedra are proportional to the squares of their edges, and consequently that *the surfaces of similar polyhedra are proportional to the squares on their edges.*

119. Again, we can prove the edges proportional to the altitudes. Draw the altitudes OF, $O'F'$, and join F, F' to any corresponding angular points A and A'.

Then since the solid angles at A, A' are equal, the inclination of OA to the base of the first polyhedron is equal to the inclination of $O'A'$ to the base of the other, so that the angles OAF, $O'A'F'$ are equal.*

$\therefore$ the triangles OAF, $O'A'F'$ are similar,

$$\text{and } AO : OF :: A'O' : O'F'.$$

Thus *the altitudes are proportional to the edges.* It follows that the *surfaces of similar polyhedra are proportional to the squares on the altitudes.*

120. *Similar solids of revolution.*

DEFINITION 23.—"Similar cones and cylinders are those which have their axes and the diameters of their bases proportional." (Euclid xi. def. 24)

This gives us at once that if h, h' be the altitudes of two cones or cylinders, and r, r' the radii of their bases, they are similar if

$$h : h' :: r : r'.$$

It is evident without proof that the bases of similar cones and cylinders are proportional to the squares of their altitudes.

Again in a cylinder, lateral surface $= 2\pi rh$.

Now in similar cylinders $\dfrac{h}{r} = \dfrac{h'}{r'}$.

$$\therefore \frac{2\pi rh}{r^2} = \frac{2\pi r'h'}{r'^2}.$$

$$2\pi rh : 2\pi r'h' :: r^2 : r'^2.$$

Thus in similar cylinders the lateral surfaces are proportional to the squares on the radii of the bases.

The student may exercise himself by deducing the same result for the cone.

All spheres may be considered similar figures. It is clear that the surfaces of spheres are proportional to the squares of the radii.

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* See chapter i. sections 51, 52.

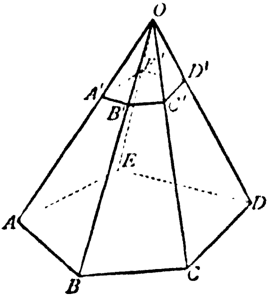
121. To sum up our results. We have proved that in all the similar solids we are concerned with

(1) Corresponding lengths are proportional.

(2) The areas of the surfaces are proportional to the squares of corresponding lengths.

Examples.—(1) A pyramid is divided, by a plane parallel to the base, into a smaller pyramid and a frustum. Prove that the smaller pyramid is similar to the whole pyramid.

Let the plane $A'B'C'D'E'$ be parallel to the base, the pyramid $OA'B'C'D'E'$ shall be similar to the pyramid $OABCDE$. The bases of the frustum are parallel and similar. (See chap. iv.) And because $A'B'$ is parallel to AB, the triangles $OA'B'$, OAB are similar. In like manner the other side-faces of the smaller pyramid can be proved similar to those of the larger pyramid.



Thus the two pyramids are contained by the same number of similar planes. And their solid angles also are equal. For instance, $\angle OA'B' = \angle OAB$; $\angle OA'E' = \angle OAE$; and $\angle E'A'B' = \angle EAB$.

$\therefore$ the solid angle at A' = solid angle at A.

$\therefore$ the pyramids are similar. Q.E.D.

Cor.: If a cone be cut by a plane parallel to the base, it cuts off a smaller similar cone.

(2) If a pyramid 9 in. in altitude is cut by a plane parallel to the base, and 5 in. above it, show that the lateral surfaces of the two parts into which the pyramid is divided are to each other as 16 : 65.

The plane will cut off a smaller similar pyramid. (See figure above.)

The altitudes of the two pyramids are 9 in. and 4 in.

$\therefore$ surface of whole pyramid : surface of smaller pyramid

$$:: 9^2 : 4^2.$$
$$:: 81 : 16.$$

$\therefore$ (dividendo).

Surface of frustum : surface of smaller pyramid

$$:: 81 - 16 : 16.$$

$\therefore$ surface of smaller pyramid : surface of frustum

$$:: 16 : 65. \quad \text{Q.E.D.}$$

(3) The altitudes of two similar cylinders are respectively 2 ft. 3 in. and 1 ft. 6 in. The lateral surface of the first is 45 sq. ft. What is the lateral surface of the second?

Let x = lateral surface of second.

$$\therefore 45 : x :: 27^2 : 18^2.$$
$$:: 9 : 4.$$
$$x = \frac{45 \times 4}{9} = 20 \text{ sq. ft.} \quad \text{Answer.}$$

EXAMPLES ON SIMILAR SOLIDS

(LENGTHS AND SURFACES)

1. The edges of two similar solids are as 7 : 9; the surface of the first is 2738 sq. ft. Find the surface of the second to three decimal places of a sq. ft.

2. The surfaces of two similar solids are as 2 : 3; the edge of the first is 37 in. Find the edge of the second to $\frac{1}{100}$ of an inch.

3. The edges of two similar solids are as 48 : 17; the altitude of the first is 10 ft. 6 in. Find the altitude of the second correct to $\frac{1}{10}$ of an inch.

4. The surfaces of two similar rectangular parallelepipeds are proportional to the squares on their diagonals.

5. The surfaces of two similar cones are proportional to the squares on their altitudes.

6. Prove that the smaller cone cut off by any section of a right cone parallel to the base is similar to the whole cone.

7. The exterior surfaces of two cubes are to each other as 2 : 7. If the edge of the first be 20 ft., find the edge of the second to one decimal place of a foot.

8. The altitudes of two tetrahedra are to each other as 7 : 23; the exterior surface of the first is 3 sq. yds. 7 ft. 53 in. Find the exterior surface of the second.

9. A pyramid 13 in. in altitude is cut by a plane parallel to the base, and 8 in. above it. Compare the lateral surfaces of the two parts into which the pyramid is divided.

10. At what altitude above the base must a parallel plane cut a pyramid so as to divide it into two parts having the same lateral surface? Altitude of pyramid = 2 ft. Answer correct to two decimal places of an inch.

11. In two similar cylinders the altitudes are 1 ft. 8 in. and 1 ft. 3 in. respectively; the lateral surface of the first is 48 sq. ft. Find the lateral surface of the second.

12. Find the whole surface of a cone, the radius of whose base is ⅓ of the radius of a similar cone whose whole surface is 243 sq. ft.

13. A cone is cut by three planes parallel to the base, at equal distances from the base and vertex. Compare the lateral surfaces of the four frustra into which it is divided.

14. The altitude of a right-angled cone is 10 in. Find at what height above the base a parallel plane must cut the cone so as to cut off a smaller cone with a lateral surface containing 256 sq. in. ($\pi = 3.1416$.)

15. Two similar cones, whose surfaces are as 1 : 8, are both cut by planes parallel to their respective bases. In the first cone the lateral surface of the cone cut off is ⅓ of the whole; in the second cone it is ⅔ of the whole. Find the ratio of the altitudes of the two cones thus cut off.

16. Find the radius of a sphere whose surface shall be 3 times that of a sphere with a radius of 6 in.

17. The diameter of a sphere is equal to the circumference of a second sphere. If the surface of the first sphere is 720 sq. ft., find the surface of the second sphere to the nearest sq. ft. ($\pi = 3.1416$.)

18. The cost of varnishing a rectangular box is 8*s.* 6*d.* What will be the cost of varnishing another box whose dimensions are all half as long again as those of the first box?

19. The cost of facing a pyramid with brick is £1023 15*s.* 9*d.* Find the cost of facing with brick a similar pyramid whose height is ⅔ that of the height of the first.

20. In a model of a church, calculated on a scale of 1 in. to 10 ft. (linear measurement), the area of a conical spire as modelled was found to be 80 sq. in. Find the area of the real spire.

21. If the cost of gilding a round ball, whose diameter is 20 in., is £2 18*s.* 4*d.*, find the cost of gilding a ball whose diameter is 32 in.

22. Find the area in sq. ft. of a terrestrial globe whose diameter is the $\frac{1}{13,904,000}$ part of the diameter of the earth, if the area of the earth's surface is 197,355,200 sq. miles. Answer to the nearest half a sq. ft.

23. The diameter of the earth being 7900 miles, and that of the moon 2160 miles, compare the areas of their surfaces; and find the radius of a sphere whose surface is equal to their sum. (*Sandhurst.*)

Section II.

122. *The volumes of similar solids are proportional to the cubes of corresponding lengths.*

This is true in the case of all solid figures. It will be sufficient for us to prove it in the case of all the solids we are concerned with.

The volume of a prism = base × altitude; and volume of a pyramid = ⅓ base × altitude.

∴ in any two similar prisms or pyramids, if V_1, V_2 be the volumes, B_1, B_2 the bases, h_1, h_2 the altitudes, and a_1, a_2 corresponding edges, we have:

$V_1 : V_2 :: h_1B_1 : h_2B_2$.

But $h_1 : h_2 :: a_1 : a_2$. (Section i.)

And $B_1 : B_2 :: a_1^2 : a_2^2$. (,,)

∴ $h_1B_1 : h_2B_2 :: a_1^3 : a_2^3$.

That is, $V_1 : V_2 :: a_1^3 : a_2^3$.

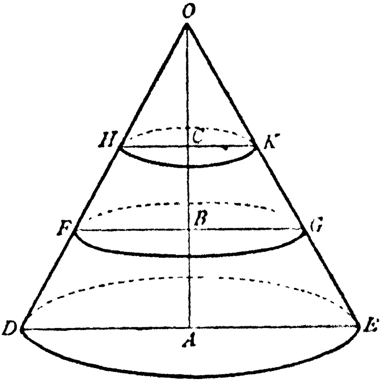
Thus the volumes of similar prisms or pyramids are proportional to the cubes of corresponding edges.

The proof will be the same in the case of the *cone* or *cylinder*, except that the result will be that the volumes are proportional to the cubes of the altitudes, or of the radii of the bases.

Finally, in the case of a *sphere*, it is clear that the volume of every sphere is proportional to the cube of the radius.

123. When two solids are similar, the smaller may be considered a *model* of the other. This will suggest the most important application of similar solids.

Examples.—(1) If a right cone on a circular base be divided into three portions by two sections parallel to the base, and at equal distances from the base and vertex, compare the three volumes into which it is divided. (*Sandhurst.*)



Let the cone ODE be cut by the two planes parallel to its base, so that the segments of the altitude OC, CB, BA are all equal.

Then the pyramids OHK, OFG are similar to the whole pyramid, and to one another.

$$\text{Volume of } OHK : \text{volume of } OFG :: OC^3 : OB^3.$$
$$:: 1 : 8.$$

$\therefore$ (dividendo) volume of OHK : volume of $HFGK :: 1 : 7$.

Again, volume of OFG : volume of $ODE :: OB^3 : OA^3$.
$$:: 8 : 27.$$

$\therefore$ volume of OFG : volume of $FDEG :: 8 : 19$.

$\therefore$ volume of OHK : volume of $FDEG :: 1 : 19$.

$\therefore$ the three volumes are to each other in the proportion:

$1 : 7 : 19$. Answer.

(2) In a model, constructed on a scale of 2 in. to 21 ft. (linear measurement), the solid content of an octagonal pillar is found to be 2·48 cubic in. Find the number of cubic ft. in the pillar.

Let x = number of cubic ft. in pillar.

$\therefore$ 2·48 cubic in. : x :: 2^3 cubic in. : 21^3 cubic ft.
$$:: 8 : 9261.$$

$$\therefore x = \frac{9261 \times 2\cdot48}{8} = 9261 \times \cdot31 \text{ cubic ft.}$$

$$= 2870\cdot91 \text{ cubic ft.} \quad \text{Answer.}$$

EXAMPLES ON THE VOLUMES OF SIMILAR SOLIDS

1. The edges of two similar solids are as 11 : 13. The volume of the first is 3 cubic yds. 13 ft. 1281 in. Find the volume of the second.

2. The volumes of two similar solids are to each other as 12,167 : 32,768. The edge of the first is 5 ft. 9 in. Find the edge of the second.

3. The surface of one solid is 6 times that of a similar solid. How many times is the first larger than the second?

4. The weights of two similar solids of the same material are 432 lb. and 540 lb. The cost of painting the surface of the first is £17 5*s*. Find the cost of painting the surface of the second.

5. The altitudes of two similar solids of the same material are as 5 : 6, and the weight of the first is 625 lb. Find the weight of the second.

6. The altitudes of two similar prisms are 12 ft. and 15 ft. Find the altitude of a similar prism whose volume is the sum of the volumes of the two first prisms.

7. The volume of a tetrahedron is 3 cubic ft. 49 in. What is the volume of a tetrahedron 3 times as high?

8. A right pyramid, whose base is a square of 7 in. a side, and whose perpendicular height is 8 in., is cut into two parts by a plane parallel to the base, and 6 in. from it. Find the volume of the two parts and their total surface. (*Sandhurst.*)

9. A pyramid is cut by two planes parallel to the base, so that the lateral surface of the three divided parts are equal. If the volume of the pyramid is 519.62 cubic in., find the volumes of these three parts. (N.B.—The roots must be taken out to four places of decimals.)

10. The height of a pyramid is 8 ft. How must a plane be drawn parallel to the base so as to divide the pyramid into two parts, whose contents shall be equal to each other? (*Sandhurst.*)

11. Prove strictly that the volumes of two similar cylinders are proportional to the cubes of their altitudes.

12. The altitude of a right-angled cone is 10 ft. Find to three decimal places of a ft. the altitude of another right-angled cone whose volume is half as large again as that of the first cone.

13. A cone is cut by a plane parallel to the base, so that the smaller cone cut off is half as large again as the remaining frustum. Find the proportion in which the cutting-plane divides the altitude.

14. Show how to find in what proportion the altitude of a cone is cut by the $n - 1$ planes parallel to the base which divide it into n equal portions.

Example: Divide a cone into three equal parts by planes parallel to the base.

15. The weights of two spheres of the same material are $85\frac{3}{4}$ lb. and $332\frac{3}{4}$ lb. If the surface of the first sphere is 154 sq. ft., find the radius of the second.

16. The edges of three similar polyhedra are 3 in., 4 in., and 6 in. Find the edge of a fourth similar polyhedron whose volume is less than that of the largest of the first three by the sum of the volumes of the other two.

17. If two cubical blocks of stone contain together 8 cubic ft., and the side of the less is to that of the greater as 3 : 4, find the side of each. (*Sandhurst.*)

18. If I pay one guinea for a cubical block of marble, of which the side is 1 ft., what ought I to pay for another cubical block of the same marble, of which the side is equal in length to the diagonal of the first block? (*Sandhurst.*)

19. The weight of the water in two cubical cisterns is as 5 : 8. The edge of the first cube is 13 in. Find to three decimal places of an in. the edge of the other.

20. A rectangular reservoir, which holds 810 gallons of water, is to have every dimension made $\frac{1}{8}$ as large again. Find how many gallons the enlarged reservoir will contain.

21. If the dimensions of a rectangular box are half as long again as those of a second, the first contains $3\frac{3}{8}$ times as much as the second.

22. The radii of two similar cylindrical blocks, one of marble, the other of stone, are as 5 : 3, while their weights are as 5 : 1.06. Compare the volumes of equal weights of marble and stone.

23. The cost of gilding a sphere is £20. Find the cost of gilding a sphere 3 times as large.

24. The diameter of Mercury is 3200 miles, that of the earth 8000 miles. How many times is the earth larger than Mercury?

25. The weights of two spheres, which are solid, and made of the same material, are 512 lb. and 729 lb. respectively. If the radius of the first sphere is 16 in., what will it cost to gild the surface of the second sphere at $1\frac{3}{4}$*d.* per sq. in.? (*Sandhurst.*)

26. An object is enlarged by a microscope 400 times. What is the apparent length of an edge .76 in. long?

27. If the surface of an object appears under a microscope 50 times larger than reality, find how many times the microscope magnifies the size of the object.

28. The model of a hall, constructed on a scale of 1 in. to 8 ft. (linear dimensions), contains 180 cubic in. Find the number of cubic ft. in the hall.

29. Find correct to a cubic ft. the content of Cleopatra's Needle, if the content of a model of it is 5.087 cubic in., given that the height of the Needle is 92 times that of the model.

30. The side of the base of a model of the Great Pyramid of Egypt is 9.55 in., and the model contains 182.785 cubic in. Deduce in cubic yds. the content of the Great Pyramid, the side of whose base is 764 ft.

CHAPTER IX.

MISCELLANEOUS PROBLEMS ON SOLIDS

[The formulæ for solid mensuration are collected here for purposes of reference. The first list contains the more important formulæ. The second list is not of such great practical importance.

In the following lists

(i.) a, b, c stand for *edges.*

(ii.) h stands for *altitudes;* h_1, h_2, h_3 for *parallel edges.*

(iii.) l for the *slant side of a cone.*

(iv.) r, r_1, r_2 for radii of circles.

(v.) B, B' for bases.

When a formula is a short one it is usually written in words, with the symbols placed after it in brackets. Sometimes a formula has to be written for the want of familiar symbols.]

124. Formulæ for the Mensuration of Solids.

PRISM

1. *Volume of rectangular parallelepiped*
   = product of three dimensions (abc).
2. *Volume of cube* (*edge* a) $= a^3$.
3. *Volume of prism = base × altitude* (Bh).
4. Also *volume of triangular prism*
   $= \frac{1}{2}$ *any side-face × perpendicular from opposite edge.*

PYRAMID

5. *Volume of pyramid* $= \frac{1}{3}$ *base* × *altitude* $(\frac{1}{3}Bh)$.

6. *Volume of frustum of pyramid* $= \frac{h}{3}\{B + \sqrt{BB'} + B'\}$

CYLINDER

7. *Lateral surface of cylinder* $= 2\pi rh$.

8. *Volume of cylinder* = *base* × *altitude* $(\pi r^2 h)$.

CONE

9. *Lateral surface of cone*
   = $\frac{1}{2}$ *circumference of base* × *slant side* (πrl).

10. *Volume of cone* = $\frac{1}{3}$ *base* × *altitude* $(\frac{1}{3}\pi r^2 h)$.

11. *Lateral surface of frustum of cone* = $\frac{1}{2}$ *slant side*
    × *sum of circumferences of bases* $(l\pi(r_1 + r_2))$.

12. *Volume of frustum of cone* $= \frac{\pi h}{3}(r_1^2 + r_1 r_2 + r_2^2)$.

SPHERE

13. *Surface of sphere (radius r)* $= 4\pi r^2$.

14. *Volume of sphere (radius r)* $= \frac{4}{3}\pi r^3$.

15. *Surface of zone or segment* $= 2\pi rh$.

16. *Volume of zone* $= \frac{\pi h}{6}\{3(r_1^2 + r_2^2) + h^2\}$.

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17. *Volume of oblique parallelepiped* $= Bh$.

18. *Frustum of right triangular prism*
    = *base* × $\frac{1}{3}$ *sum of parallel edges* $\left(B \times \frac{h_1 + h_2 + h_3}{3}\right)$.

19. *Volume of tetrahedron* (edge a) $= \frac{a^3}{6\sqrt{2}}$.

20. *Lateral surface of frustum of cylinder* $= \pi r(h_1 + h_2)$.

21. *Volume of frustum of cylinder* $= \pi r^2\left(\frac{h_1 + h_2}{2}\right)$.

22. *Volume of segment of sphere* $= \frac{\pi h}{6}(3r_1^2 + h^2)$.

125. Problems are often set in solid mensuration of greater or less complexity, which cannot be solved by the aid of one formula only, but require two or three. These may be roughly grouped under the four following heads:

(*a*) When the problem is simply to compare the surfaces or volumes of two different solids. For instance, it may be asked what portion should be cut off a cylinder to equal in volume a pyramid of given dimensions. This case may be called *comparison of solids.*

(*b*) When some substance of measurable quantity is transferred from one solid receptacle to another. For instance, water may run through cylindrical pipes into a rectangular reservoir; or gunpowder may be taken from a cylindrical canister to fill a spherical shell; or *the material itself* of some solid may be used to form another solid, as when a square bar of iron is hammered out into cylindrical wire, and so on. This case may be called *measurement of one solid by means of another.*

(*c*) When a solid figure is formed by the union of two or more simple solids. For instance, a cylindrical tower may be capped by a hemispherical dome; the roof of a square-built house may be in shape a prism; a tunnel may be a rectangular parallelepiped surmounted by a semi-cylinder, and so on. This may be called *addition of solids.*

(*d*) When one solid is described about or inscribed in another solid, or, more generally, when one solid is conceived as hollow, and having another inserted in it. Thus a cone may be described about a sphere; or a sphere inscribed in a cylinder; or, in practice, a spherical body may be thrown into a conical vessel; or, which is practically the same case, shot may be piled in the form of a pyramid. This case may be called *subtraction of solids.*

Examples follow illustrating the four cases.

Examples.—CASE (*a*).—(1) Find the difference in content of two tumblers, one of which is cylindrical (height $4\frac{1}{4}$ in., diameter $2\frac{5}{16}$ in.), while the other is an inverted frustum of a cone (height $3\frac{3}{8}$ in., diameter of top $2\frac{3}{4}$ in., diameter of bottom $2\frac{1}{4}$ in.); and find how much water each will hold, if a gallon contains $277\frac{1}{4}$ cubic in.

Volume of cylindrical tumbler $= \pi \left(\frac{37}{32}\right)^2 . \frac{17}{4}$ cubic in.

$$= \pi . \frac{1369}{32^2} . \frac{17}{4}.$$

Content of first tumbler in pints $= \frac{\cancel{8} \times \cancel{4}}{1109} \times \frac{\pi \times 1369 \times 17}{\cancel{32} \times 32 \times 4}.$

$$= \frac{\overset{11}{\cancel{22}} \times 1369 \times 17}{7 \times 1109 \times 32 \times \underset{2}{\cancel{4}}} = \frac{256003}{496832}.$$

$= \cdot 515$ pint.

Volume of conical tumbler $= \pi . \frac{31}{24} \left\{\left(\frac{17}{16}\right)^2 + \frac{17}{16} . \frac{23}{16} + \left(\frac{23}{16}\right)^2\right\}.$

$$= \pi . \frac{31}{24} . \frac{1209}{16^2}.$$

Content of second tumbler in pints $= \frac{8 \times 4}{1109} \times \pi \times \frac{31}{24} \times \frac{1209}{16^2}.$

$$= \frac{\overset{11}{\cancel{22}} \times 31 \times \overset{403}{\cancel{1209}}}{7 \times 1109 \times \underset{8}{\cancel{24}} \times \underset{4}{\cancel{8}}} = \frac{137423}{248416}.$$

$= \cdot 553$ pint.

∴ the second tumbler holds more by ·038 pint. Answer.

(2) A cylinder, whose height = diameter of base, has a surface equal to a cube. Compare their volumes.

Let r = radius of base of cylinder, a = edge of cube.

$\therefore$ surface of cylinder $= 2\pi r\,(h + r) = 2\pi r \,.\, 3r.$

$= 6\pi r^2.$

Surface of cube $= 6a^2.$

$\therefore\ 6a^2 = 6\pi r^2.$

$a = \sqrt{\pi} \,.\, r.$

Again, volume of cylinder $= \pi r^2 \,.\, h = 2\pi r^3.$

Volume of cube $= a^3.$

$\therefore$ ratio required $= 2\pi r^3 : a^3.$

$= 2\pi r^3 : \pi^{\frac{3}{2}} r^3.$

$= 2 : \pi^{\frac{1}{2}}.$

The answer may be left in this form. If worked out, it will be found to be: $1\cdot 128 : 1.$

CASE (*b*).—(3) A bucket is in shape a conical frustum (height = 9 in., diameters of top and bottom surface = 10 in. and $7\frac{1}{2}$ in. respectively). Find how much lower the water will stand in a well whose diameter is 5 ft., after the bucket has been filled 24 times.

$$\text{Volume of bucket} = \frac{\pi \cdot 9}{3}\left\{5^2 + 5 \cdot \tfrac{15}{4} + \left(\tfrac{15}{4}\right)^2\right\} \text{ cubic in.}$$

$$= 3\pi \cdot \tfrac{25}{16}\,(16 + 12 + 9) = \frac{3\pi \cdot 25 \cdot 37}{16} \text{ cubic in.}$$

$$\therefore \text{ content of 24 buckets} = \frac{3\pi \cdot 25 \cdot 37 \cdot \overset{3}{\cancel{24}}}{\underset{2}{\cancel{16}}} \text{ cubic in.}$$

$$= \frac{3\pi \cdot 25 \cdot 37 \cdot 3}{2} \text{ cubic in.}$$

Let h = amount the water will sink.

$\therefore$ h is the altitude of a cylinder, the radius of whose base = 2 ft. 6 in.

$$\therefore \text{ volume of cylinder} = \pi \cdot 30^2 \cdot h \text{ cubic in.}$$

$$\therefore \pi \cdot 900 \cdot h = \frac{3\pi \cdot 25 \cdot 37 \cdot 3}{2}.$$

$$h = \frac{\cancel{3}\pi \cdot \cancel{25} \cdot 37 \cdot \cancel{3}}{2 \cdot \underset{\underset{4}{\cancel{36}}}{\cancel{900}} \cdot \pi} = \frac{37\pi}{8\pi} = 4\tfrac{5}{8} \text{ in.} \quad \text{Answer.}$$

(4) A solid metal sphere, 6 in. in diameter, is formed into a tube 10 in. in external diameter, and 4 in. in length. Find the thickness of the tube. (*Sandhurst.*)

Let x = thickness of the tube.

$$\therefore \text{ volume of tube} = \text{difference of two cylinders.}$$

$$= \pi \cdot 4\left\{5^2 - (5 - x)^2\right\}.$$

$$\text{Also volume of sphere} = \frac{4\pi}{3} \cdot 3^3 = 36\pi.$$

$$\therefore 4\pi\left\{5^2 - (5 - x)^2\right\} = 36\pi.$$

$$25 - (5 - x)^2 = 9.$$

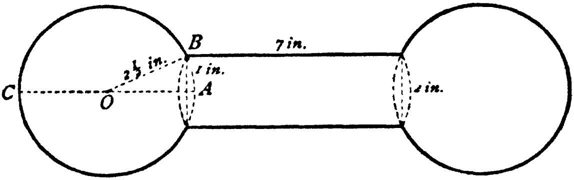
$$(5 - x)^2 = 16.$$

$$x - 5 = \pm 4.$$

$$x = 5 \pm 4 = 9 \text{ or } 1.$$

The smaller value is the thickness of the tube. The larger value really gives the remainder of the external diameter when the thickness of the tube has been subtracted once from it.

Case (*c*).—(5) **Find the weight of a dumb-bell consisting of two spheres of 5 in. diameter, joined by a cylindrical bar 7 in. long, and 2 in. in diameter, an iron ball 4 in. in diameter weighing 9 lb.** (*Sandhurst.*)



The dumb-bell = sum of two equal segments of a sphere, and a cylinder.

We must first find the altitude of one of the segments.

$$\text{Altitude } CA = CO + OA.$$

$$= 2\tfrac{1}{2} + \sqrt{(2\tfrac{1}{2})^2 - 1}.$$

$$= \frac{5 + \sqrt{21}}{2} \text{ in.}$$

If we take $\sqrt{21} = 4\cdot58$, altitude $= 4\cdot79$ in.,

$$\therefore \text{ volume of either segment} = \frac{\pi \times 4\cdot79}{6}\left\{3 \,.\, 1^2 + (4\cdot79)^2\right\}.$$

$$= \frac{\pi}{6} \times 4\cdot79 \times 25\cdot9441.$$

$$= \pi \times 20\cdot712 \text{ cubic in. (nearly).}$$

$$\text{Volume of cylinder} = \pi \,.\, 1^2 \,.\, 7 = 7\pi \text{ cubic in.}$$

$$\therefore \text{ total volume of dumb-bell} = \pi \times 41\cdot424 + 7\pi.$$

$$= \pi \times 48\cdot424 \text{ cubic in.}$$

Again, volume of iron ball $= \frac{4}{3}\pi \,.\, 2^3 = \frac{32\pi}{3}$ cubic in.

$$\therefore \text{ 1 cubic in. of iron weighs } 9 \times \frac{3}{32\pi} = \frac{27}{32\pi} \text{ lb.}$$

$$\therefore \text{ weight of dumb-bell} = \frac{27}{32\pi} \times \pi \times 48\cdot424.$$

$$= \tfrac{27}{32} \times 48\cdot424 \text{ lb.}$$

= 40 lb. 14 oz. (to nearest ounce). **Answer.**

(In questions of this kind it is often necessary to examine the data carefully. Thus a careless reading might give the student the impression that he had to find the sum of a cylinder and two spheres, instead of a cylinder and two segments of a sphere.)

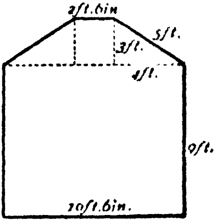
(6) A regular octagonal room, whose side is 10 ft. 6 in., is surmounted by a pyramidal ceiling, which in its turn is capped by an octagonal lantern whose side = 2 ft. 6 in. The height of the side-walls is 9 ft., and the slant edge of the pyramidal ceiling stretching from the side-wall to the lantern is 5 ft. Find the cost of painting both walls and ceiling, at 4¾*d.* per sq. ft.

The lower part of the room is an octagonal prism.

The upper part, as far as the lantern, is a frustum of a symmetrical octagonal pyramid.

The lantern, of course, will not be painted.

Hence we have to find the lateral surface of a prism (= 8 rectangles), and add on to it the lateral surface of the frustum of a pyramid (= 8 trapezoids). In fact, we have to find 8 times the value of the annexed figure, representing one side of the room.



$$\text{Area of rectangle} = 10\tfrac{1}{2} \times 9 = \tfrac{189}{2} \text{ sq. ft.}$$

The altitude of trapezoid is easily seen to be 3 ft.

$$\therefore \text{ area of trapezoid} = \tfrac{13}{2} \times 3 = \tfrac{39}{2} \text{ sq. ft.}$$

$$\therefore \text{ area of surface to be painted} = 8 \times \left(\tfrac{189}{2} + \tfrac{39}{2}\right).$$

$$= 8 \times 114 \text{ sq. ft.}$$

$$\text{Cost of painting} = 8 \times 114 \times 4\tfrac{3}{4}d.$$

$$= \overset{2}{\cancel{8}} \times 114 \times \frac{19}{\cancel{4}} = 361s.$$

$$= £18\ 1s. \quad \text{Answer.}$$

CASE (*d*).—(7) The side of the base of a hexagonal prism is 4 in., and the altitude is 9 in. Find the volume of an inscribed cylinder.

The base of the inscribed cylinder will be the circle inscribed in the base of the hexagon.

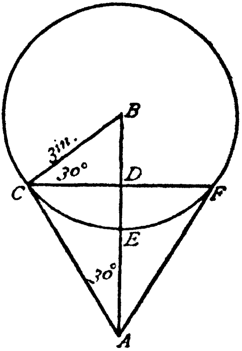
$$\therefore\ r = \frac{\sqrt{3}}{2} \cdot 4 = 2\sqrt{3} \text{ in.}$$

$$\text{Volume of inscribed cylinder} = \pi\,(2\sqrt{3})^2 \cdot 9.$$
$$= 12 \times 9\pi = 108\pi \text{ cubic in.}$$
$$= 339 \text{ cubic in. (to nearest cubic in.)} \quad \text{Answer.}$$

(8) A tangent cone is drawn to a sphere whose radius is 3 in. If the vertical angle of the cone is 60°, find the volume of the space included between the sphere and the cone.

Let the figure represent a plane section of the sphere and cone, passing through the centre *B* of the sphere, and the vertex *A* of the cone.

Then we have to find the difference between the volumes of the cone and the segment of the sphere *CEF*.



$$\text{Volume of cone} = \frac{\pi}{3} \cdot CD^2 \cdot AD.$$

$$\text{Volume of segment of sphere} = \frac{\pi}{6} \cdot DE\,(3CD^2 + DE^2).$$

The angle $CAB = \frac{1}{2}$ angle $CAF = 30°$, and angle $ACB = 90°$.

$$\therefore \text{ angle } CBD = 60°, \text{ and angle } BCD = 30°.$$

$$CD = CB \cos 30° = \frac{3\sqrt{3}}{2} \text{ in.}$$

$$AD = CD \cot 30° = \frac{3\sqrt{3}}{2} \times \sqrt{3} = \frac{9}{2} \text{ in.}$$

$$DE = BE - BD = 3 - CB \sin 30° = 3 - \frac{3}{2} = \frac{3}{2} \text{ in.}$$

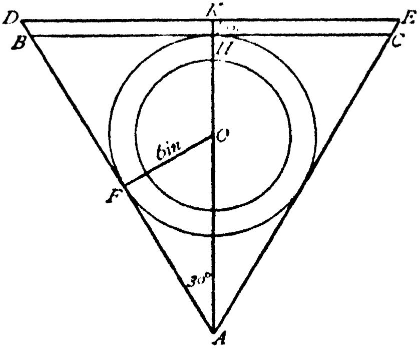
$$\therefore \text{ volume of cone} = \frac{\pi}{3} \cdot \frac{27}{4} \cdot \frac{9}{2} = \frac{81\pi}{8} \text{ cubic in.}$$

Segment of sphere $=\frac{\pi}{6} \cdot \frac{3}{2}\left(3 \cdot \frac{27}{4}+\frac{9}{4}\right)=\frac{45\pi}{8}$ cubic in.

$\therefore$ volume included $=\frac{\pi}{8}(81-45)=\frac{36\pi}{8}=\frac{9\pi}{2}$ cubic in.

If $\pi=3.1416$,
Volume $=14.137$ cubic in. Answer.

(9) A hollow shell 12 in. in diameter is placed in a conical vessel, whose vertical angle is 60°, and water poured into it until it just covers the shell and fills the cavity in it, when the shell, emptied of the water in it, is removed, and a solid ball of the same diameter substituted for it. The water stands ½ in. above it. Find approximately the thickness of the shell. (*Sandhurst.*)



Let the figure represent a plane section of cone and shell, drawn as in the last example.

Then the frustum *DBCE* of the cone *DAE* represents the amount of water requisite to fill the shell.

It will readily be seen that the triangles *ABC*, *ADE* are equilateral, and that *F*, *H*, *K* are points of bisection for their sides.

Volume of frustum *DBCE*

$$=\frac{\pi}{3} \cdot HK\left\{DK^2+DK \cdot BH+BH^2\right\}.$$

Now $HK=\frac{1}{2}$ in. . . . . (1).

$BH=BF=AF=OF \cot 30^\circ=6\sqrt{3}$ in. . . . (2).

and $DK : KA :: BH : HA$.

But $HA=OA+OH=OF \operatorname{cosec} 30^\circ+OH=3OH$.
$=18$ in.

$\therefore KA=18\frac{1}{2}$ in.

and $DK : \frac{37}{2} :: 6\sqrt{3} : 18$.

$$DK=\frac{37\sqrt{3}}{6}.=\frac{37}{2\sqrt{3}} \text{ in.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

$$\therefore \text{ volume of frustum} = \frac{\pi}{3} \cdot \tfrac{1}{2}\left\{\frac{37^2}{12} + \frac{37}{2\sqrt{3}} \cdot 6\sqrt{3} + 108\right\} \text{ cubic in.}$$

$$= \frac{3997\pi}{72} \text{ cubic in.}$$

Let r = interior radius of shell,

$$\therefore \tfrac{4}{3}\pi r^3 = \frac{3997\pi}{72}.$$

$$\therefore r^3 = \frac{3997}{72} \times \tfrac{3}{4} = \frac{3997}{96} \text{ cubic in.}$$

Whence we shall find $r = 3{\cdot}47$ cubic in. (very nearly).

$\therefore$ thickness of shell $= 6 - 3{\cdot}47 = 2{\cdot}53$ in. **Answer.**

MISCELLANEOUS EXAMPLES ON SOLIDS

(a) Comparison of Solids

1. Show that a right cone, a hemisphere, and a cylinder, all of which have the same base and altitude, have their solid contents as the numbers 1, 2, 3. (*Sandhurst.*)

2. Two thin vessels, without lids, each contain a cubic ft. The one is a rectangular parallelepiped on a square base, whose height is half its length; the other a right circular cylinder whose height is equal to the radius of its base. Compare the amounts of material which it would require to make them, the thickness being the same for each. ($\pi = 3.1416$.) (*Sandhurst.*)

3. At what height must a plane parallel to the base cut a pyramid, whose base is 400 sq. ft. in area, and whose height is 60 ft., so that the frustum cut off below it may be equal in volume to the frustum of a cylinder, the diameter of whose base is 28 ft., and whose greatest and least heights are $19\frac{1}{11}$ ft. and $3\frac{7}{11}$ ft.?

4. Which wine-glass contains most—a hemispherical one whose diameter at the top is $3\frac{1}{2}$ in., or a conical one whose diameter at top, and height, are both equal to $3\frac{1}{2}$ in.?

5. A Dutch cheese is spherical in shape, a Stilton cylindrical. Find the height of a Stilton cheese which stands on a base 6 in. in diameter, and is equal in content to a segment of a Dutch cheese $4\frac{1}{3}$ in. thick, the diameter of the Dutch cheese being $13\frac{2}{3}$ in.

6. Two buckets have the same depth, i.e. 8 in. One is cylindrical, the radius of its base being 5 in., and the other is a conical frustum, the radii of its ends being 6 in. and 4 in. Which holds the greatest amount of water, and by how much? Answer to three decimal places of a pint.

7. A conical block of stone is 42 ft. high, and 5 ft. 6 in. in radius of base. Find the altitude of a frustum of it which shall be equal in bulk to a rectangular stone block whose dimensions are 12 ft., 9 ft. 6 in., and 8 ft. 8 in.

8. What must be the area of the base of a cylinder whose altitude is 12 in., if its lateral surface is $\frac{1}{8}$ that of a prism on a regular hexagonal base, whose side is $5\frac{1}{8}$ in., and the altitude of which is 8 in.?

9. A square pyramid is 5 times as high as a cylindrical column 30 ft. high. The volume of the pyramid is 2,000,000 cubic ft., and the volume of the column is 4620 cubic ft. By how much will the pyramid stand on a greater area than 250 equal columns of the above size?

10. How deep must the water be in a rectangular reservoir whose length is 150 ft., and breadth 45 ft., in order that the volume of water may be the same as in a cylindrical one in which the depth of the water is 10 ft., and the diameter of the base 90 ft.? ($\pi = 3.1416$.) Answer to three decimal places of a foot.

11. The floor of a room is a regular hexagon whose side is 9 ft. Find the height, correct to the nearest quarter of an inch, in order that the room may contain as many cubic ft. of air as a rectangular room 20 ft. 3 in. long, 10 ft. 8 in. broad, and 8 ft. 10 in. high.

12. A frustum of a cylinder and a frustum of a cone are on equal bases, and the portions of the axes intercepted in each are equal. If the volume of the conical frustum $= \frac{13}{36}$ of that of the cylindrical frustum, prove that the diameters of the top and bottom surfaces of the conical frustum are to each other as 1 : 12.93.

13. A sphere, and a cylinder whose altitude is 3 times the radius of its base, have equal surfaces. Show that the volume of the sphere is 1.257 times that of the cylinder.

14. A cube is 3 times a tetrahedron. Show that edge of cube is to edge of tetrahedron in the ratio of the side of a square to its diameter.

15. The volume of a right-angled cone is half that of a hemisphere. Show that the whole surface of cone is .8047 times that of the hemisphere.

16. A cone and a cylinder have the same altitude, and the radii of their bases are as 3 : 2. Show that their volumes are in the ratio 3 : 4.

17. A sphere and a square pyramid, all whose edges are equal, have the same volume. Show that the surface of the pyramid is 1.48 times the surface of the sphere.

18. An officer proceeding to India in a troop-ship is allowed 60 cubic ft. of baggage. A lieutenant had two cases and a basket, one case being 4 ft. 2⅙ in. long, 2 ft. 2¾ in. wide, and 2 ft. 4¼ in. deep, and the other being half those dimensions, and the cylindrical basket being 3½ ft. high by 2 ft. in diameter. By how much did he exceed or fall short of his allowance, actual contents being taken for the cylinder, and fractions of a cubic ft. being omitted in the result? (*Sandhurst.*)

19. If a 12-lb. shot be 2.17 in. in radius, find the weight of a cubical safe of iron whose side is 3 ft., and thickness of metal ¾ in. Answer correct to a lb.

20. The altitude of a right-angled cone is 10 in. Find the edge of a cube of equal volume, to the nearest tenth of an inch. ($\pi = 3.1416$.)

21. A spherical iron shell, whose external and internal diameters are 6 in. and 4 in., weighs 22 lb. 6 oz.; and a cylindrical leaden pipe 10 in. long, whose external and internal diameters are 6 in. and 4 in., weighs 64 lb. 13 oz. Find to two decimal places how many times lead is heavier than iron.

22. The altitude of a cylinder is 12 in., and the diameter of base 10 in.; the base of a cone is 3 times that of the cylinder, and its altitude half that of the cylinder. Find the radius of a sphere whose volume = the sum of the volumes of cone and cylinder.

23. If the altitude of a frustum of a cone = difference between the radii of the ends, the volume = $\frac{1}{4}$ the volume of a hollow shell whose inner and outer diameters are the same as those of the ends of the conical frustum.

(b) Measurement of one Solid by another

24. Water is flowing at the rate of 10 miles an hour, through a pipe 15 in. in diameter, into a rectangular reservoir 187 yds. long, and 84 yds. wide. Calculate the time in which the surface will be raised 1 in. (*Sandhurst.*)

25. A rectangular reservoir, whose length and breadth are 125 ft. and 63 ft., is filled by a cylindrical pipe (diameter = $3\frac{1}{2}$ in.), through which the water runs at the rate of 8 miles per hour. Find how much the surface will be raised in 10 hours, and also what number of gallons will be poured in during that time.

26. The water is $8\frac{1}{2}$ ft. deep in a rectangular reservoir whose base covers an area of 5280 sq. yds. In what time can the water be emptied by a pipe 6 in. in diameter, through which the water runs at the rate of 18 miles an hour?

27. Find to $\frac{1}{10}$ of a mile at what rate per hour the water must run through a pipe 3 in. in diameter, in order to fill in 24 hours, to the depth of 6 ft., a rectangular reservoir whose length is 220 ft., and breadth 72 ft.

28. A cylindrical reservoir, whose diameter is 40 ft., is filled by a pipe whose cross-section is a square (side = 4 in.). If the water runs at the rate of 10 miles per hour through the pipe, find to $\frac{1}{10}$ of a minute how long it will take to raise the surface 1 ft.

29. The water stands 7 ft. 6 in. deep in a rectangular reservoir whose base covers an area of 6240 sq. yds. If it is emptied by a pipe whose diameter is $4\frac{1}{2}$ in., through which the water flows at the rate of 12 miles per hour, find approximately in tons what weight of water will remain in the reservoir after the pipe has been running 8 hours.

30. A bucket is in shape a conical frustum whose height is 10 in., and the diameters of whose ends are 9 in. and 12 in. Find how much lower the water will stand in a well whose diameter is 3 ft. 9 in. after the bucket has been filled 27 times.

31. Given the well and bucket as in the last example. Find how many times the bucket must be filled in order to make the water sink $24\frac{2}{3}$ in.

32. How many conical wine-glasses, each $2\frac{1}{2}$ in. in height, and 2 in. in diameter at the top, can be filled from a hemispherical bowl whose diameter is 10 in.?

33. How many tumblers, each of which is in shape an inverted conical frustum (height $3\frac{3}{4}$ in.; diameter of top and bottom $2\frac{1}{2}$ in. and 2 in.), can be filled from a cylindrical water-jug, whose height is 12.2 in. and diameter 5 in.?

34. A box, whose internal dimensions are 4 ft., 3 ft., and 1 ft. 6 in., is half full of oats. How many times may a round measure 4 in. deep and 22 in. in circumference be entirely filled from it?

35. A rectangular box, whose length and breadth are 2 ft. 6 in. and 1 ft. 9 in., has corn thrown into it 20 times from a round measure, whose depth is 5 in. and diameter 1 ft. Find to the nearest in. how high the corn will stand in the box if distributed evenly.

36. Eighteen conical piles of corn are each 4 ft. high and 8 ft. in diameter at the base. If this corn be distributed evenly over the floor of a granary 40 ft. long and 25 ft. broad, find to the nearest half in. to what height it will reach.

37. Gunpowder is taken from a full cylindrical canister, whose height is 10 in., and inner diameter 3 in., to fill a spherical shell, whose interior diameter is 4 in. Find how much gunpowder is left in the canister if 32 cubic in. of gunpowder weigh 1 lb. Answer to two decimal places of a lb.

38. How many spherical shells, whose interior diameters are each 3 in., can be filled with gunpowder from a cylindrical canister whose height is 1 ft. 4 in., and the diameter of whose base is 6 in.?

39. A solid metal sphere, 4 in. in diameter, is formed into a tube 1 ft. in external diameter, and 3 in. in length. Find the thickness of the tube to three decimal places of an in.

40. A bar of iron, in shape a triangular prism, 3 ft. long, whose ends are right-angled triangles, the sides being 7 in. and 4 in., is melted down and cast into a cannon-ball. Find the diameter of the cannon-ball to the nearest hundredth of an in.

41. A cylindrical bar of lead, the diameter of whose cross-section is $2\frac{1}{2}$ in., and whose length is 9 in., is melted down and cast into bullets, each $\frac{1}{2}$ in. in diameter. How many bullets can be so formed, and what will be the weight of each to three decimal places of an oz., if 1 cubic in. of lead weighs 6.6 oz.?

42. A rectangular bar of iron, whose dimensions are 9 in., $4\frac{1}{2}$ in., and 3 in., is drawn out into wire 270 ft. long. Show that its thickness is less than $\frac{8}{9}$ in.

43. A cubical mass of iron, whose edge is $5\frac{1}{2}$ in., is drawn out into wire $\frac{1}{10}$ in. thick. Find its length. ($\pi = 3.1416$.)

44. A rectangular bar of brass, whose length is 13 in., and whose cross-section is a square (side = 4 in.), is drawn out into wire 187 yds. long. Find the weight of 99 yds. of this wire, if 1 cubic ft. of brass weighs 8500 oz.

45. The diameter of a cylindrical iron rod is $1\frac{1}{8}$ in. What must be its length, if the iron composing it can be melted and cast into a spherical shell 5 in. and 3 in. in external and internal diameters?

46. A mass of gold can be beaten into gold-leaf extending over 10,000 sq. yds., and $\frac{1}{40000}$ in. thick. Find to the nearest yd. what length of wire $\frac{1}{20}$ in. thick could be made out of the same mass of gold. ($\pi = 3.1416$.)

47. A spherical mass of iron, whose radius is 4⅓ in., is drawn out into wire $\frac{1}{12}$ in. in diameter. Find its length in yds.

48. A rectangular bar of iron, 5½ sq. in. in area of cross-section, is formed into a spherical shell whose exterior diameter is 7 in., and which holds sufficient gunpowder to fill a cylindrical canister, 5 in. in diameter of base, to a depth of 3⅓ in. Find the length of the bar of iron to $\frac{1}{10}$ in.

(c) Addition of Solids

49. An iron boiler is in the form of a circular cylinder, 9 ft. long, with hemispherical ends. Its extreme diameter is 3 ft., the metal is 1 in. thick, and the weight is 3623⅛ lb. What is the weight of a cubic in. of iron? (*Sandhurst.*) (N.B.—Apparently the cylinder is 9 ft. long *without* the ends. The result suggests some error in the question. All the Sandhurst questions are given exactly as set.)

50. A cylindrical boiler is terminated by a hemisphere at each end. Its total length is 8 ft. 8 in., and its extreme breadth 4 ft. 2 in. Find the weight of the iron composing it, if the metal is 1 in. thick, and a cubic in. of iron weighs 4.2 oz.

51. If a room be 40 ft. long by 20 ft. broad, and contain 12,800 cubic ft., what addition will be made to its cubic contents by throwing out a semi-circular bow at one end? ($\pi = 3.1416$.) (*Sandhurst.*)

52. The dimensions of a room are: breadth 12 ft., and height 9 ft. Find what addition to its cubical content could be made by throwing out a semicircular bow at one end.

53. A cylindrical tower 24 ft. in diameter, and 30 ft. high, is capped with a hemispherical dome. The top of the dome is cut off, and over the orifice formed is built a cylindrical lantern 8 ft. in diameter, and 10 ft. high, closed at the top by a plane surface. Find the total exterior surface of the building. (*Sandhurst.*)

54. A round tower 120 ft. high is surmounted by a conical top 40 ft. high. If the diameter of the tower be 60 ft., find the whole exterior surface. ($\pi = 3.1416$.)

55. An observatory consists of a round tower capped by a hemisphere. The total height is 60 ft., and the external diameter of the tower is 35 ft. If the brickwork composing it is 14 in. thick throughout, find the volume of the bricks in cubic ft.

56. Find the whole exterior surface of a house which is square-built, with the ordinary sloping roof on both sides. Total height = 50 ft., height of the side-walls = 41 ft., length = 60 ft., breadth = 24 ft.

57. A square tower, the side of whose base is 14 ft., and whose height is 50 ft., is terminated by a pyramidal spire with 4 sides, and 24 ft. in height. Find the total exterior surface in sq. ft.

58. A hexagonal room is capped by a pyramidal roof. If a side of the hexagon is 10 ft., the total height 13 ft., and height of the walls 8 ft., find (1) the cost of papering the walls and ceiling at 1*s.* 9*d.* per sq. yd., allowing for a window 6 ft. by 3 ft. on 4 out of the 6 sides, and a door of the same size on the other two; (2) the cubic content of the room to the nearest cubic ft.

59. A building is in shape a cylinder surmounted by a hemisphere, which in turn is surmounted by a conical lantern. The height of the cylinder is 30 ft., and the diameter of the base 24 ft.; the height of the lantern is 4 ft., and the diameter of the base of the lantern 6 ft. Find the total exterior surface to $\frac{1}{10}$ of a cubic ft., if all the measurements given are external. ($\pi = 3.1416$.)

60. A room is in shape rectangular, with a semicircular bow at either end of the length. The extreme length, breadth, and height are to each other as 3 : 2 : 1. If the room contains 5000 cubic ft., find its dimensions, each correct to an inch.

61. The cross-section of a tunnel consists of a rectangle surmounted by a semicircle. If the total height is 35 ft., and breadth 20 ft., find approximately how many cubic ft. of air there are in a mile's length of tunnel. ($\pi = \frac{22}{7}$.) Find also, by taking $\pi = 3.1416$, how many cubic ft. the former answer is greater than the exact number.

62. The cross-section of a brick subway 20 ft. long is a rectangle surmounted by a semicircle. The total height and breadth (both exclusive of the bricks) are 8 ft. and 4 ft., and the thickness of the bricks is 4½ in. Find the weight of the bricks, if a brick containing $\frac{9}{16}$ of a cubic ft. weighs 5 lb.

63. A gasometer consists of a cylinder with a spherical top (segment of sphere). If the circumference is 6⅔ poles, and the spherical portion rises 6 ft. above the cylindrical, find what must be the total height of the gasometer in order that it may contain 40,000$\frac{9}{14}$ cubic ft. of gas.

64. How many tons of water does a river pour into the sea per hour, supposing that the section of the river is a rectangle, whose breadth is 80 ft., with a semicircle below it, and that the total depth is 45 ft., the stream running at the rate of 3 miles an hour?

65. Find the weight of a solid cylindro-conical projectile, the whole length being 12 in., and the length of the conical part being 4 in., and the diameter of the circular end being 3 in., if a cubic in. of iron weighs 4.2 oz.

66. A top is an inverted cone surmounted by a cylinder. Find the number of cubic in. it contains, if its total length is 4 in., length of conical part 2¼ in., and diameter of top surface 3½ in.

67. A solid is composed of a hemisphere surmounted by a cone. The radius of the hemisphere is 2.25 in., and the height of the cone 3.65 in. Find the exterior surface of the solid. ($\pi = 3.1416$.)

68. A cylindrical rod has conical ends, each 2 in. in length. If the diameter of any cross-section of the cylinder is 2⅝ in., and the total length of the rod is 1 ft. 4 in., find the weight of the rod, 1 cubic in. of the material of which it is composed weighing 2½ oz.

69. The top surface of a table is a regular octagon, the greatest diagonal of which is 32 in., and the thickness is ¾ in. There are four cylindrical legs reaching to a height of 2 ft. 4 in. from the ground each, and 1½ in. in diameter,

and these legs are joined together by four rectangular cross-pieces of wood, each 18 in. long, 2 in. broad, and ½ in. thick. Find to the nearest cubic in. the solid content of the table.

70. A cylindrical column stands on a base which is in shape a conical frustum. The diameter of the base of the frustum is 3 ft., the diameter of the base of the column is 1 ft. 9 in., the height of the column without the base is 12 ft., and the height of the base is 1 ft. Find total volume of column and base. Answer to the nearest cubic ft.

71. A conical funnel ends in a circular tube. If the diameter of the top of the funnel is 8 in., and that of the tube ½ in., while the whole funnel is 1 ft. long, and the tube 2 in. long, find to four places of decimals the amount of error in assuming that the funnel holds 5 pints. ($\pi = 3.1416$.)

72. A solid, consisting of a right cone standing on a hemisphere, is placed in a right cylinder full of water, and touches the bottom. Find the weight of water displaced, having given that the radius of the cylinder is 3 ft., and its height 4 ft., the radius of the hemisphere 2 ft., and the height of the cone 4 ft. ($\pi = 3.1416$.) (*Sandhurst.*) [N.B.—This example belongs also to Case (*d*).]

(d) Subtraction of Solids

73. If a rectangular parallelepiped on a square base be inscribed in a cylinder, whose diameter is 2 ft., and height 10 ft. 6 in., find its volume.

74. The base of a prism 8 ft. long is a regular hexagon, the side of which is 6 in. Find the lateral surface of a circumscribed cylinder. ($\pi = 3.1416$.)

75. Find the surface of a sphere which can be described about a cube whose edge is 4 in. Answer to the nearest sq. in.

76. The altitude of a cylinder is 10 in., and the diameter of the base is 6 in. Find the volume of the inscribed prism on a regular hexagonal base. Answer to nearest cubic in.

77. The altitude of a symmetrical pyramid on a hexagonal base, whose side is 7 in., is 2 ft. Find the lateral surface of the circumscribing cone.

78. The altitude of a cone is 2 ft. 3 in., and the diameter of the base is 15 in. Find the volume of the space included between it and its inscribed square pyramid.

79. A cone is reduced by trimming to a symmetrical pyramid on a hexagonal base. Find to two decimal places what portion of the material is removed.

80. A square pyramid, whose altitude is 8 in., and side of base 6 in., is reduced by trimming to a cone. Find to the nearest cubic in. the volume of the material to be removed.

81. A round pillar, whose height is 20 ft., and diameter of cross-section 18 in., is reduced by trimming to a regular octagonal shape. Find how much it loses in weight, if 1 cubic ft. of the stone composing it weighs 170 lb.

82. If a cylindrical rod, 34 in. long, and $\frac{1}{2}$ sq. in. area at its end, rest cross-wise in a rectangular vessel, whose length and breadth are 18 and 16 in. respectively, find approximately the weight of water which must be poured in so as just to reach the highest point where the rod touches the side.

83. A rectangular tank is 5 ft. long by 2 ft. broad. Find to two decimal places of an in. how much the water will rise if a sphere of 10 in. radius be totally submerged in it. ($\pi = 3.1416$.)

84. Find the diameter of a spherical body which, being thrown into a cylindrical tank whose diameter is 4 ft., will make the water rise 3 in.

85. A number of equal iron balls, forming a complete pile on a triangular base, the lowest layer consisting of 15 balls, weighs 3 cwt. Find the diameter of each ball. 1 cubic ft. of iron weighs 450 lb. (*Sandhurst.*)

86. A complete pile of iron shot is arranged on a square base, there being 8 shot in the side of the lowest layer. If the diameter of each shot is 6 in., find to the nearest cwt. the total weight of the shot, if 1 cubic ft. of iron weighs 7776 oz.

87. A hollow right prism stands upon a base which is an equilateral triangle. The vertical faces of the prism are squares, the side of a square being 1 ft. The prism is filled with water, and the largest possible sphere is then submerged in it. Find the amount of water remaining in the prism. ($\sqrt{3} = 1.73$.) (*Sandhurst.*)

88. A cylindrical vessel, the diameter of whose base is 6 in., has the largest possible sphere submerged in it, and is then filled up with water so as just to cover the sphere. The sphere is then taken out. How high will the water stand in the vessel?

89. A spherical body fits exactly into a cylindrical box, the inner diameter of which is 8 in. If the upper part of both cylinder and sphere is cut away by a plane passing through a point in the top circumference of the cylinder, and also a point where sphere and cylinder touch, find the content of the remainder of the cylinder which is not taken up by the remainder of the sphere. Find also the lateral surface of the part of the box remaining.

90. An inkstand is in shape a conical frustum, the diameters of whose top and bottom surfaces are 2 in. and 4 in., and whose height is $1\frac{7}{8}$ in. The well for the ink is cylindrical, takes up the whole of the top surface, and is $1\frac{1}{8}$ in. deep. Find to three decimal places of a cubic in. the number of cubic in. in the glass part of the inkstand.

91. The total depth of a conical wine-glass is 3 in., and the vertical angle is 60°. It is filled with water, and a cylindrical body is inserted vertically in it, resting finally on the sides of the glass when 1 in. in perpendicular height from the bottom. Find the amount of water remaining in the glass to $\frac{1}{10}$ of a cubic in.

92. A hollow shell of the greatest external diameter possible is placed in a cylindrical vessel whose diameter at the base is 8 in., and then the vessel is completely filled with water, including the cavity of the shell. The shell, with its cavity empty, is then taken out, and the water sinks $\frac{2}{3}$ in. Find the internal diameter of the shell to the nearest hundredth of an inch.

93. A hemispherical bowl, whose internal radius is 1 ft., is filled with water, and placed on a horizontal table. In the water there is placed, with its vertex touching the centre of the bottom of the bowl, and its axis vertical, a cone whose vertical angle is 90°. Find the amount of water left in the bowl after the intrusion of the cone. (*Sandhurst.*)

94. A vessel 10 in. in height is in shape a segment of a sphere greater than a hemisphere. A cone, whose vertical angle is 60°, is placed in the vessel, and is found to rest exactly on both sides of the top surface. Find to a cubic in. the volume of water which the vessel will hold after the intrusion of the cone. ($\pi = 3.1416$.)

95. From a point 2 ft. from the surface of a sphere, whose radius is 4 ft., a tangent cone is drawn to the sphere. Find the exterior surface of the cone to a sq. in. ($\pi = 3.1416$.)

96. Show how to find the radius of a sphere inscribed in a cone of known base and altitude. In a conical wine-glass, whose depth is 4 in., and diameter of top 6 in., there is immersed a spherical body, the top of which is flush with the surface of the water when the wine-glass is entirely filled. Find the radius of the sphere, and what fraction of the volume of the wine-glass is occupied by it.

97. A conical vessel, whose depth is to the diameter of its top surface as 2 : 3, is filled with water to the depth of 2 in., and a sphere, whose radius is $\frac{3}{4}$ in., is then submerged in it. How high does the water stand above the sphere?

98. A sphere of radius 3 in. is placed in a conical vessel whose vertical angle is 60°, and then water is poured in so as to completely cover the sphere. How much will the water sink in the vessel when the sphere is removed?

99. A wine-glass is an inverted frustum of a cone, the radius of its base being $\frac{1}{2}$ in. A sphere of diameter 2 in. is thrown into the glass, and touches the bottom and the sides at the same time. The glass is then filled with water so as to just cover the sphere. Find how much water will be left in the glass if the sphere be removed. ($\pi = 3.1416$.)

100. A spherical body 4 in. in diameter is thrown into an inverted cone, the vertical angle of which is 120°. Find the distance between any point at which the cone and sphere touch, and the vertex. Find also to three decimal places of a cubic in. the volume included between the cone and the sphere.

101. A hollow paper cone, whose vertical angle is 60°, is held with its vertex downwards, and in it there is placed a sphere of radius 2 in. The portion of the cone remote from the apex is then cut away along the line where the paper touches the sphere. Find the exterior surface of the body thus formed. (*Sandhurst.*)

102. A conical wine-glass, whose vertical angle is a right angle, is filled with water. A hemisphere of radius 1 in. is immersed in the water with its convexity downwards. It is found that when it rests on the sides of the wine-glass its flat surface is flush with the level of the water. Deduce the depth of the wine-glass, and find the amount of water left in it after the immersion of the hemisphere. ($\sqrt{2} = \frac{7}{5}$.) (*Sandhurst.*)

APPENDIX TO BOOK II.

ON THE REMAINING SOLID FIGURES

126. Besides the polyhedra we have discussed, there can be formed an infinite variety of polyhedra contained by any number of polygons. Of *regular polyhedra*, however, there are only five. This limitation depends on the fact that *the plane angles by which a solid angle is contained must be less than four right angles.*

Now to form a regular polyhedron, three equal angles at least must meet at each angular point; and as the sum of these angles must be less than 360°, each of these angles is less than 120°. Hence the faces of a regular polyhedron must be equilateral triangles, squares, or regular pentagons; for the angle of a regular hexagon is 120°, and that of a regular polygon of more than 6 sides is greater still. Bearing in mind, therefore, that the angles at each vertex must be less than 360°, we shall see that the only possible regular polyhedra are those which have their solid angles formed by the meeting of:

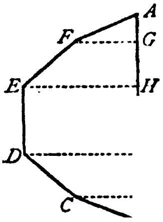
(1) Three equilateral triangles (sum of angles $= 3 \times 60^\circ = 180^\circ$).
(2) Three squares (,, $= 3 \times 90^\circ = 270^\circ$).
(3) Four equilateral triangles (,, $= 4 \times 60^\circ = 240^\circ$).
(4) Three regular pentagons (,, $= 3 \times 108^\circ = 324^\circ$).
(5) Five equilateral triangles (,, $= 5 \times 60^\circ = 300^\circ$).

In all other cases the angles will be equal to or greater than 360°. These five solids have 4, 6, 8, 12, 20 faces respectively, and are known by the names of the *cube*, the *tetrahedron*, the *octahedron*, the *dodecahedron*, and the *icosahedron*. The first three we have met with.

127. A *solid of revolution* was defined as the solid figure described by the revolution of a plane figure round a fixed straight line called its axis. We have only investigated three, but we can show that the results we have obtained are capable of being extended to others.

128. *To find the volume of a solid of revolution generated by a polygon revolving round one of its sides which is fixed.*

Let AB be the axis round which the polygon revolves. Draw the perpendiculars FG, EH, DK, CL. Then AFG, as it revolves, traces out a cone. So also CLB, while FH, EK, DL trace out frusta of cones.



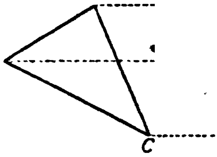
Thus the volume of the solid = the sum of a number of cones and frusta of cones.

If any side be parallel to the axis, the corresponding frustum will of course be a cylinder.

129. *To find the volume of a solid of revolution generated by a triangle revolving round an axis outside it.*

From the angular points of the triangle draw perpendiculars to the axis.

∴ triangle ABC = trapezium AF + trapezium BE − trapezium CD.



Now each trapezium, as it revolves, generates a frustum of a cone. Hence the volume of the solid = sum of two frusta − the third frustum.

Since every polygon can be divided into triangles, this method can be extended to the case in which the revolving figure is any polygon.

130. *Solid Ring.* If the revolving figure is a circle, the resulting solid is called a *solid ring.* A solid ring may therefore be defined as a figure described by the revolution of a circle round a fixed straight line in the same plane, called the axis, which does not cut the circle. The circle traced out by the centre of the revolving circle is called the *length.* Any section of the ring made by a plane passing through the axis is a circle, which is called a *cross-section.* Since a solid ring may be considered a cylinder bent round so as to meet, we have the results:

Surface of ring = length × circumference of cross-section.

Volume of ring = length × cross-section.

ANSWERS TO EXAMPLES

Book I.

CHAPTER I.

Section I. (Page 11.)

(a)

1. 10 ft.
2. 96 ft.
3. 55 ft.
4. 120 yds.
5. 58 miles.
6. 280 ft.
7. $\dfrac{l}{\sqrt{2+2\sqrt{5}}}$.
8. 100 ft.
9. 149·9437 ft.
10. 420·04 ft.
11. 76·75718 ft.

(b)

12. 16·323 ft.
13. 6 ft. 8 in.
14. 250 ft.
15. BC $= \dfrac{100 \cos 10° 45' \sin 5° 30'}{\sin 5° 15'}$.

    Height $= \dfrac{100 \sin 10° 45' \sin 5° 30'}{\sin 5° 15'}$.
16. 73·20 ft.
17. 25 ft.
18. 48 ft.
19. 114·41 ft.

    $\{= 25(\sqrt{10}+\sqrt{2})$ ft.$\}$
21. $\frac{1}{2}$ mile.
22. $15+3\sqrt{6}$ *or* $15-3\sqrt{6}$ ft.
23. $10(2 \cos 20° 16' \pm \sqrt{1-4 \sin 20° 16'})$, both values of which are $+$*ve*
24. a.
25. Length $= \dfrac{2q}{\sqrt{\cot^2 \frac{\beta}{2} - \cot^2 \frac{\alpha}{2}}}$.

    Altitude $= \dfrac{q \cot \frac{\alpha}{2}}{\sqrt{\cot^2 \frac{\beta}{2} - \cot^2 \frac{\alpha}{2}}}$.
26. Height $= \dfrac{a \sin \alpha \cos \beta}{\sin (\alpha - \beta)}$.

    Distance $= \dfrac{a \cos \alpha \cos \beta}{\sin (\alpha - \beta)}$.
27. 3107·82 ft.
28. 229·6 yds.
29. 3041 ft.
30. $\dfrac{a \sin \theta \sin \phi}{\sin (\theta + \phi)} = 44·1726$ ft.

Section II. (Page 20.)

1. 323 sq. yds. 7 ft. 55 in.
2. 10 chains.
3. 10·392 sq. ft. ($6\sqrt{3}$ sq. ft.)
4. 48 ft.; 84 ft.
5. 84 sq. in.
6. 15 acres.
7. 84.
8. 210.
9. 20·976 sq. in.
10. 56·78 sq. in.
11. 7 ft. 7 in.
12. 1 ft. 8 in.; 1 ft. 9 in.
13. 5·264 in.
14. 300 sq. ft.
15. 43·3 sq. in. 13·16 in.
16. 16 ft.
17. 5 a. 10 p. $27\frac{1}{2}$ sq. yds.
18. 3 a. 3 r. 35 p. (nearly)
    (= $15\sqrt{7}$ sq. ch.);
    6 ch. $13\frac{1}{2}$ yds. $\left(\frac{5\sqrt{7}}{2}\text{ ch.}\right)$
19. 25 chains.
20. 11 yds. 2 ft. $0\frac{1}{2}$ in.
21. 324 yds. (to nearest yd.)
    $\left\{= 110\left(\frac{2+\sqrt{21}}{\sqrt{5}}\right)\text{ yds.}\right\}$
22. £3 6*s.*
23. First pays £2 5*s.* 9*d.*;
    second pays £2 6*s.* $2\frac{3}{4}$*d.*
24. £91 5*s.*
25. £2 7*s.* $0\frac{6}{11}$*d.*
26. 336 yds.
27. 2 r. 19 p. $29\frac{1}{4}$ sq. yds.
28. £1 0*s.* 2*d.*
29. 36528·18 sq. yds.
30. 19301·51 sq. yds.

CHAPTER II.

Section I. (Page 29.)

1. 3 fur. 8 po.
2. 1 mile 385 yds.
3. £1195 0*s.* $8\frac{1}{2}$*d.* (correct to a farthing.)
4. 22·077 ft.
5. 20 ft.
6. 1583 yds. 2 ft. 9 in.
7. 34 ft.
8. 214 yds. 2 ft.
9. $8\sqrt{57}$ in.; 8 ft. 8 in.
10. 1·035 ft.; 2·07 ft.
11. 4 in.; 5 in.
12. 1 ft. 5 in.
13. $\sqrt{3}-1$; $\sqrt{3}+1$.
14. 5 ft.; 4·8 ft.
16. 4 in.; $4\sqrt{3}$ in.
17. 2 ft.; 10 in.
18. 1 ft. 8 in.; 1 ft. $2\frac{1}{2}\frac{1}{8}$ in.
19. 1 ft. 5 in.; 1 ft. 4 in.
20. 132° 4′ 30″; 47° 55′ 30″.
21. 15 ft. 1 in.
23. 15 in.; $4\sqrt{10}$ in.
24. 5 yds.
25. 15 yds.
26. 1 ft. 9 in.
27. 2 ft.; 2 ft. 6 in.
28. 5 ft.
29. $64\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right) = 33$ ft. $1\frac{1}{2}$ in.
30. 14 ft. 8 in.

Section II. (Page 42.)

(a)

1. 40 acres.
2. 1 sq. mile 340 acres.
3. 30 a. 2 r. 20 p.
4. 308 yds.
5. 4 a. 0 r. 36 p.
6. £40 6*s.* 8*d.*
7. £5 2*s.* 8$\frac{1}{2}\frac{1}{4}$*d.*
8. £2 18*s.* 8*d.*
9. 891 sq. yds.
10. 9$\frac{1}{4}$ chains.

(b)

11. 30 sq. yds. 2 ft.
12. 3 a. 1 r. 15 p. 22$\frac{1}{4}$ sq. yds. 6 sq. ft.
13. 41 ft. 3 in.
14. 99 yds.; 198 yds.
15. 22464.
16. 432 yds.
17. 132 sq. in.
18. £2.
19. £3 4*s.* 11*d.*
20. £6 10*s.* 2$\frac{1}{2}$*d.*
21. £1 9*s.* 2*d.*
22. 11 ft. 3 in.
23. £3 4*s.* 9$\frac{1}{2}$*d.*
24. £2 2*s.*
25. 18*s.* 6$\frac{2}{3}$*d.*
26. 11 ft. 4 in.
27. £5 16*s.* 8*d.*
28. £2 8*s.* 4*d.*
29. 1*s.* 3*d.*
30. 6 yds.
31. £600 12*s.*
32. 11,880; £462.
33. 832 sq. yds.
34. £41 6*s.* 10$\frac{1}{2}$*d.*
35. 110 yds.; 55 yds.
36. 10 yds.
37. £180.
38. £24 10*s.*
39. 78 yds.; 58 yds.
40. 6 yds.; 1 yd.

(c)

41. 349 sq. yds. 1 sq. ft. 69 sq. in.
42. 3 sq. ft. 127 in.
43. 11 yds. 2 ft. 8 in.
44. 10 ft.
45. 39 poles; 65 poles.
46. 1 ft. 6 in.; 7 ft. 8 in.
47. 15·78 in.; 23·66 in.
48. 2 a. 3836·82 sq. yds.

(d)

49. 8 sq. ft. 108 in.
50. $\sqrt{6}+\sqrt{2}$ sq. yds.
51. 1 a. 3 r.
52. 110 yds.
53. 2·58 acres.
54. 1860 sq. ft.
55. $\frac{1}{8}$ of the lawn.
56. $7\frac{39}{128}$ sq. in.
57. 8.
58. No. of whole panes, 60; no. of half-panes, 24; area of each, 18 sq. in.

(e)

59. $\frac{1}{2}$ acre.
60. 50 sq. ft.
61. 207 sq. in.
62. 6 a. 3 r.
63. 1 a. 2 r.
64. 228 sq. ft.
65. £1 13*s.* 4*d.*
66. 140 yds.
67. 176 yds.
68. 7·013939 sq. in.

(f)

69. 1 a. 2 r. 32 p.
70. 2 ft. 11 in.
71. 3 a. 1 r. 32 p.
72. 1$\frac{1}{2}$ acres.

73. 75 sq. in.
74. 7 sq. ft. 84 in.
75. 15 sq. yds. 4 ft. 48 in.
76. 26 sq. yds.
77. 1 r. 38 p.
$\{=\frac{3}{4}(\sqrt{3}+\sqrt{23})$ sq. ch.$\}$
78. 10,800 sq. yds.; 2 a. 0 r. 37 p. $0\frac{3}{4}$ yds.

CHAPTER III. (Page 55.)

1. 1 a. 465 yds.
2. 2 a. 400 sq. yds.
3. 4 a. 0 r. 16 p.
4. 72 sq. yds.
5. 728 sq. ft.
6. 214 sq. yds. 6 sq. ft.
7. 1096 sq. yds.
8. 4 a. 1 r. 7 p.
9. 1 a. 2 r. 4 p.
10. 163·28 sq. in.
11. 1692 sq. yds. 6 sq. ft.
12. 1 a. 0 r. 10 p. 20 sq. yds.
13. 109 sq. ft.
15. 12·855 sq. in.
16. 2 sq. ft. 43 in.
17. 4 a. 2 r. 26 p.
18. Area = 110 sq. yds. 6 ft.; $AB = 65$ ft.
19. 2 sq. ft.
20. Two solutions:
(1) AE produced 50, or $37\frac{1}{2}$ yds.
(2) CD produced 10, or $13\frac{1}{3}$ yds.
21. 38 sq. yds. 2 ft. 6 in.
22. 35 sq. in.
23. $\frac{25\sqrt{3}}{108}a^2$.
24. 270 sq. ft.
25. 53·706 sq. in.
26. 7 a. 1120 sq. yds.
27. 9 a. 3 r. 38 p. $10\frac{1}{8}$ sq. yds.
28. 19 a. 3 r. 3 p. $19\frac{1}{4}$ sq. yds.
29. 12 a. 2 r. 35·68 p.
30. 7 a. 0 r. 27·632 p.

CHAPTER IV.

Section I. (Page 65.)

1. 20.
2. 24.
3. 8; 4.
4. 4.
5. 11·55 ft. $\left(\frac{20}{\sqrt{3}}\right)$
6. 2·9 ft.
7. 9 ft.
8. 29·856 ft.; 30·909 ft.
9. 43·38838 in.
11. 3·708 ft.
13. $4 : 3\sqrt{3}$.
14. $1 : \sqrt{3}$.
15. 1·25 : 1.
16. 36·96 ft.
17. 3·2359 ft. $\left\{\frac{a(\sqrt{5}+1)}{2}.\right\}$
18. 33·54 in. $(15\sqrt{5}.)$
19. 2·618 : 1.
$\left\{=5a\left(\frac{3-\sqrt{5}}{2}\right) : 5a.\right\}$
20. 1 ft. 8·78 in. $(\sqrt{3}$ ft.)
21. ·59 : 1.
22. 24.
23. 1·748 : 1. $(4\sqrt{2} : \sqrt{5}+1.)$
26. $\frac{2}{3}$ of the radius of first circle.

Section II. (Page 72.)

1. 172 sq. in.
2. 166 sq. yds. 2 ft. 70·8 in.
3. 14 sq. ft. 55 sq. in.
4. 15 sq. ft. 110 in.
5. 11 sq. ft. 135 in.
6. 115·754 sq. in.
7. ·12 in.
9. 104 sq. ft.
10. 1·55 : 1.
12. 1110 links.
13. 190 sq. in.
14. 3·141576 : 1.
16. $\frac{1}{3}$ of the former.
17. 30 chains.
18. £10 1*s.* 2*d.*
19. £4 6*s.* $7\frac{3}{4}$*d.*
20. £9 10*s.* $6\frac{3}{4}$*d.*
21. 724 sq. in.; 107·6 in.
22. 53 sq. ft.
23. Base = 16·24 sq. ft.; square base = $14\frac{1}{16}$ sq. ft.; octagonal base = 16·98 sq. ft.
24. £11 5*s.*
25. 37·8875 sq. ft. $\left(\frac{175\sqrt{3}}{8}.\right)$
26. 1835 sq. yds. 3 ft.
27. A regular polygon with 180 sides; area = 10,312 sq. yds. 2 ft.
28. 3 a. 3810 yds.
29. 390 yds.

CHAPTER V.

Section I. (Page 93.)

(a)

1. 42 ft. 5 in.; 5 ft.
2. 1 mile 4 fur. 13·2312 po.; 15·915 ft. ($\pi = 3\cdot1416$.)
3. ·00126; $1\frac{1}{128}$ fur.
6. 22,400 times.
7. 18,368 times.
8. $15\frac{1}{2}$ miles.
9. 2 ft. 11 in.
10. ·02 in.
13. 14·645 in.
14. 10 ft.
15. 8*s.* $8\frac{1}{2}$*d.*
16. £12 8*s.* $9\frac{6}{7}$*d.*
'**17.** 97·045 yds.
18. 510.

(b)

19. 691 yds. (correct to a yd.)
20. 435·734 miles.
21. ·6981 in.; ·6976 in.
22. 5 min. to 8.
23. At 20 and $45\frac{5}{11}$ min. past 6.
24. $2^\circ\ 10'\ 7''\cdot5$.

(c)

25. 5 ft.
26. $\frac{1}{2}$ in. or $4\frac{1}{2}$ in.
27. 3·38 in.
28. $10\frac{3}{4}$ in.
29. 1 ft. 9 in.
30. 1 ft.

Section II. (Page 86.)

(a)

1. 107 sq. yds. 8 ft. $99\frac{1}{4}$ in.; 107 sq. yds. 8 ft. 43 in.
2. 154 sq. in.
3. 3 chains 87 links.
4. 9 ft. 4 in.
5. 2 chains 20 links.
7. $2\sqrt{2} : 3\sqrt{3}$.
8. $2 : \sqrt{3}$.

9. 1 : 1·06.
10. 31·32 sq. in.
12. 12 ft.
13. 11·46; 9; 6·93 sq. in.
14. 49,327,538 sq. miles.
15. 1089.
16. 201 nails.
17. 1 a. 3 r. 17 p.; £9 14*s*. $10\frac{7}{8}$*d*.; £7 13*s*. $1\frac{11}{16}$*d*.
18. $987\frac{4}{7}$ sq. yds.
19. 15*s*. $0\frac{3}{4}$*d*.
20. 42 sq. yds.
21. 88 sq. in.; 3 ft. 8 in.
22. 3 sq. ft. 112 in.
23. 98 sq. in.
24. $\frac{1}{3}$ mile.
25. $113\frac{1}{7}$ sq. yds.
26. 16 a. 2134 sq. yds.
27. £4 16*s*. 5*d*.
28. £4 11*s*. 8*d*.
29. 6·93 in. $(4\sqrt{3})$; 9·80 in. $(4\sqrt{6}.)$
30. $12\sqrt{5}$, $12\sqrt{10}$, $12\sqrt{15}$, $24\sqrt{5}$ in.
31. 169 coins; $78\frac{2}{9}$ sq. in.; string $=60\frac{4}{21}$ in.; circumference $=62\frac{6}{7}$ in.
32. $40\frac{7}{32}$ sq. in.; 3·265 sq. in.
33. 745 sq. ft. $(+\frac{18}{38}.)$
34. 387 sq. yds. $8\frac{1}{4}$ ft.
35. 18·97 ft.
36. 14 yds.; 42 yds.
37. 1·0272 sq. ft. $\{\frac{1}{2}(3\sqrt{3}-\pi).\}$

38. Series $=r^2(\pi-2)\left(1+\frac{1}{2}+\frac{1}{4}\ \ldots\ldots\right)=2r^2(\pi-2)$; 2·2832 sq. ft.

39. Series $=\left(\pi-\frac{3\sqrt{3}}{2}\right)r^2\left(1+\frac{3}{4}+\frac{9}{16}+\ \ldots\ldots\right).$

40. Series $=\left(\frac{3\sqrt{3}}{2}-\frac{3\pi}{4}\right)a^2\left(1+\frac{3}{4}+\frac{9}{16}+\ \ldots\ldots\right).$

40*a*. Put $a=r$, and add together the two last answers.

(b)

41. 19·328 sq. ft.
42. 8 ft. 3 in.
43. ·1566 sq. ft.; 12·4098 sq. ft.
44. 106°.
45. 1 sq. ft. 126 in.
46. 489 sq. ft.; 64 ft. (approx.)
47. 500·4 sq. in.
48. 7 ft. 1 in.
49. 3·46 sq. in.
50. 1·85 sq. in.
51. 10 in.
52. 3 sq. ft. 59 in.
53. 7·7376 sq. in. $\{=24(2\sqrt{3}-\pi)$ sq. in.$\}$
55. 44·2224 sq. in. $\{=6(4\pi-3\sqrt{3}).\}$
56. 1·7168 sq. in. $\left\{=2r^2\left(1-\frac{\pi}{4}\right).\right\}$
57. 116·84 sq. in.
58. 7·902 sq. in.
59. 19 ft. 2 in.; 106 sq. ft. 16 in.
60. 37·7 sq. in. $(12\pi.)$

CHAPTER VI.

Section I. (Page 93.)

1. 135·92 ft.
2. 106 ft.
3. 150 ft.
4. 20 ft. 4 in.
5. If $x = BC$ in feet, then $x^2 + 140x = 4000 \tan\alpha \cot\beta$.
6. 94 ft. 4 in.
7. 40 ft.
8. 2000 yds.
9. 5 ft. 10 in.; 4 ft. 1 in.; 1 ft. 10$\frac{1}{6}$ in.
10. 26 ft. from *B*; 21 ft. 8 in. from *C*.
11. 4$\frac{1}{3}$ ft.
12. 1 ft. 5 in.
13. 40 yds. 2 ft. 3 in.
14. 1 ft. 4 in.; 2 ft. 2$\frac{3}{4}$ in.; 1 ft. 10$\frac{1}{4}$ in.; 10$\frac{3}{4}$ in.
15. 5 ft.
16. 10$\frac{3}{4}$ in.; $\frac{1}{48}$ in. to a mile.
17. $\frac{5}{24}$ in. to a mile; 28$\frac{1}{8}$ miles.
18. 9 m. 6 fur. 32 po.; 1 m. 1 fur. 34 po.
19. 469 miles.
20. $\frac{3}{1000}$.
21. 3 in. to 1000 miles.
22. ·6609375 in.; 440 m. 5 fur.
23. £1 1*s.* 6*d.*
24. $\frac{1}{16}$ in. to a mile.
25. 25 ft. × 13 ft. 4 in.; $\frac{1}{8}$ in. to the mile.

Section II. (Page 99.)

3. 150 yds., 200 yds., 250 yds.; 15,000 sq. yds.
4. 4 : 9.
5. 400 bushels.
6. £20 13*s.* 7$\frac{1}{5}$*d.*
7. 4$\frac{1}{2}$ acres.
8. 150 yds.
10. 43$\frac{3}{4}$ sq. yds.
11. 5·773 in. $\left(\frac{10}{\sqrt{3}}\right)$;

    2·391 in. $\left(\frac{\sqrt{2}-1}{\sqrt{3}} . 10 \text{ in.}\right)$;

    1·835 in. $\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}} . 10 \text{ in.}\right)$
12. 1 : ·414 : ·318 : ·268.
13. £38 9*s.* 4*d.*; £11 17*s.* 4*d.*
14. 32·42 sq. in.
15. 8·88 sq. in.
16. $\frac{1}{6,423,183}$ (nearly).
17. 1 in. to 10 ft.; 363 sq. ft.
18. 4·33 in.; 4·04 in.
19. 1200 sq. miles; 4·014 sq. in.
20. ·1897 in.; ·024 sq. in.
21. 11 a. 1960 yds.
22. 1·69 sq. in.
23. 3 a. 1 r. 14·03648 p.
    3 a. 1 r. 14·9376 p.
    3 a. 1 r. 14·487 p.
24. 78125 : 1; $\frac{1}{400}$ sq. in.; 976$\frac{9}{16}$ sq. in.
25. 110 miles.
26. 2·12 sq. in.
27. 70·4 in.
28. 3 : 1.
29. 4·976 times.
30. 3 ft. 3·633 in.

Book II.

CHAPTER II.

Section I. (Page 110.)

1. 73 ft. 10 in.
2. 14 ft. 7 in.
3. 42 sq. yds. 4 ft. 104 in.
4. $4\frac{1}{2}$ sq. ft.
5. 13 ft. 5 in.
6. 102 ft.
7. 10 ft.
8. 172 sq. yds. 7 ft.
9. 88 sq. ft.
10. 9·219 ft.
11. 5 ft.
12. 9 yds.; 12 yds.; 21 yds.
14. 4·87 in.
15. 22 sq. ft.
16. 164·877 sq. in.
17. 2 ft.
18. 472 sq. ft.
19. $27\frac{2}{9}$ sq. ft.
20. 9 sq. in.
21. 7*s.* $8\frac{1}{2}$*d.*
22. $7\frac{7}{15}$*d.*
23. $1\frac{1}{2}$ ft.
24. 44 sq. ft. $1\frac{1}{2}$ in.
25. 4·7320508 sq. ft. $(3+\sqrt{3}.)$

Section II. (Page 117.)

(a)

1. 39 cub. ft. 1529 in.
2. 1782 cub. ft.
3. 107 cub. ft. 297 in.
4. 168 yds.
5. 38 yds.
6. 28 sq. yds. 8 ft. 6 in.
7. 8 cub. in.;
   1122·3 cub. in. $(648\sqrt{3}.)$
8. 3, 5, 7 in.; 27; 125;
   343 cub. in.
9. 2 ft.
11. 12 ft. 7·2 in.
12. 3·3 cub. ft.
13. 21 gallons.
14. 500 lb.
15. 1 ft. 6 in.; 2 ft.
16. $307\frac{1}{3}$ cub. in.
17. 15 tons 19 cwt. 3 qr. 17 lb.;
   £18 13*s.* $2\frac{1}{4}$*d.*
18. (1) 17 cwt. 1 qr. 12 lb.
   (2) 14 cwt. 2 qr. 14 lb. (most accurate.)
   (1) is obtained by considering volume as = surface × thickness of iron;
   (2) by considering it the difference of two cubes.
19. 6·52 in.
20. 143604·1 oz.

(b)

21. 256 cub. yds. 21 ft. 576 in.
22. 33 cub. yds. 23 ft. 108 in.
23. 5 cub. yds. 9 ft.
24. 2 ft. 10 in.
25. 29 ft.
26. 12 sq. ft. 132 in.
27. 252 yds.
28. 11·51388 cub. ft.;
   8·594208 cub. ft.

29. 120 cub. in.
30. 33 lb. 8·1 oz. (avoirdupois.)
31. 55 cub. yds. 26 ft. 1056 in.
32. 7 ft. 6 in.
33. 8 ft. 4 in.
34. £44 2*s.*
35. 106,902.
36. 162 cub. yds.
37. 14 hrs.
38. 3 ft.
39. 3,000,000 sq. ft.
40. $\frac{1}{400{,}000}$ in.
41. £67 14*s.* 2*d.*
42. 10 ft. 8 in.; 1½ in.
43. 48 sq. ft. 54 in.
44. £6.
45. 7·8 times.
46. 10 cwt. 1 qr. 23 lb. 14 oz.
47. 3 in.
48. 1755 gallons.
49. 2113 gallons.
50. 4⅛ in.
51. 15 cub. ft.
52. 90 sq. ft.
53. 1 hr. 44 min.
54. 12 hrs. 16¼ min.; 5 hrs. 37½ min.
55. 5·7 in. (nearly.)
56. 2·8 miles per hour.
57. 103,680 gallons.
58. Surface = 13 sq. ft. 77·7 sq. in.; volume = 1 cub. ft. 221·7 cub. in.

(Observe that as the pipes must meet one another, they are not exactly parallelepipeds in shape.)

(c)

60. 105 cub. yds. 23 ft. 1080 in.
61. 13 ft. 6 in.
62. 1 ft.
63. 46 sq. yds. 2 ft.
64. 198 sq. yds. 2 ft.
65. 18 cub. ft. 396 in.
66. 15 ft. 9 in.
67. 2 sq. ft. 88 in.
68. 123 yds. 8 in.
69. 38⅔ gallons.
70. 190,200 cub. ft.; 281 yds. 2 ft. 4 in.

CHAPTER III.

Section I. (Page 131.)

1. 420 sq. ft.
2. 4 sq. ft. 96 in.
3. 145·49 sq. ft.
4. 12 sq. ft.
5. 15·464 sq. ft.
6. 8¼ sq. ft.
7. 10 ft.
8. 6·832 sq. ft.
9. 2 ft. 9·7 in.
10. 12·536 ft.
11. 149·9 sq. in.
12. 43·3 sq. ft.
13. 8·79 ft.
14. 200 sq. yds. 6⅔ ft.
15. 16*s.* 8*d.*
16. £17 12*s.* 11*d.*
17. 30 sq. ft.

Section II. (Page 137.)

1. 2893 cub. yds. 5 ft. 1296 in.
2. 11 cub. yds. 18 ft.
3. 4 cub. yds. 2 ft.
4. 76 cub. yds. 5 ft. 576 in.
5. Surface = 11·196 sq. ft.; volume = 2·598 cub. ft.
6. 117 cub. yds. 22 ft. 899 in.
7. 1166⅔ cub. ft.

8. 7·2 in.
9. 4·107 ft.
10. 13 ft. 6 in.
11. $514\frac{1}{2}$ cub. ft.
12. 382 cubic in.
13. 190 cub. yds. $3\frac{1}{3}$ ft.
14. $21\frac{1}{4}$ cub. ft.
15. $10\frac{1}{2}$ cub. ft.
17. 200 tons.
18. 7 tons 11 cwt. 2 qr. $26\frac{5}{27}$ lb.
19. 72 ft.
20. 5 tons 7 cwt. 1 qr. 11 lb.
21. 8582 gallons.
22. 4 ft. 10 in.
23. 7 miles per hour. (6·997 m.)
24. 255,563 gallons.
25. ·417 in.
26. 42 ft.
27. 10,602 cub. ft.
28. 8 ft.
29. (1) 2·598 cub. ft.
    (2) 16·2375 gallons.
30. £657 18*s.*

CHAPTER IV.

Section I. (Page 146.)

1. 43·27 in.; 33 sq. ft. 113 in.
2. 896 sq. ft.
3. 49·066 sq. in.
4. 2 in.; 2·236 in. ($\sqrt{5}$) (two); 2·57 in. $\left(\sqrt{\frac{11+\sqrt{5}}{2}}\right)$ (two).
5. 108 sq. in.
6. 4 sq. ft. 89 in.
7. 28·618 sq. ft. (each face = $3\sqrt{91}$ sq. ft.)
8. 4 in.
9. 4·8 ft.
10. 6·964 in.
11. 9·27 in.
12. 3 sq. ft. 49 in.
13. 13·416 ft. ($6\sqrt{5}$); 18 ft.
14. 142 sq. yds. 2 ft.
15. 12·7 ft.; $4171\frac{1}{2}$ sq. yds.
16. £16 13*s.* 4*d.*
17. £22 4*s.* $5\frac{1}{3}$*d.*
18. 15 ft.
19. £1719 10*s.*
20. 240 sq. yds.
21. £297 19*s.* $1\frac{3}{4}$*d.*
22. £660.
23. $151\frac{1}{9}$ sq. yds.; £2 16*s.* 8*d.*
24. $22\frac{3}{4}$ sq. in.
25. 250 sq. in.; 9·6 in.

Section II. (Page 156.)

1. 7943 cub. yds. 11 ft. 864 in.
2. 2143 cub. ft. 1296 in.
3. $623\frac{1}{2}$ cub. ft. (nearly.) $\left(=\frac{500\sqrt{14}}{3}.\right)$
4. 4955 cub. ft. (3600 cot 36°.)
5. 306 cub. ft. 1704 in.
6. 18 cub. ft. 896 in.
7. 7 cub. ft. 448 in.
8. (1) $42\frac{3}{8}$ sq. ft.; (2) 4 ft. 8 in.; (3) 3 ft. 4 in.
9. 4 yds. 1 ft. 4 in.
10. 764 ft.; 3,458,939 cub. yds.
11. 249·4 cub. ft. (The plane must divide the altitude in the proportion 1 : ·26.)
12. Vol. frustum
    $= 128\frac{3}{8}$ cub. in.;
    vol. pyramid $= 2\frac{1}{24}$ cub. in.

$$\begin{array}{lr} & \text{sq. in.} \\ \text{Surface frustum} = & 166{\cdot}7 \\ \quad ,, \quad \text{pyramid} = & \underline{10{\cdot}7} \\ \text{Total surface} = & 177{\cdot}4 \end{array}$$

13. 18·279 cub. in.
14. 12·728 cub. in.
16. 15·793 cub. in.
17. 58·93 cub. ft.
19. 4 cub. ft. 1278 in.
20. 72 cub. yds. 16 ft.
21. 91 cub. yds. 3 ft.
22. 630 cub. yds. 14 ft.
23. 2·449 in. ($\sqrt{6}$);
7·425 cub in. $\left(\frac{21\sqrt{2}}{4}.\right)$
24. 16 tons 13 cwt. 1 qr. 21 lb.
25. £17 16*s*.
26. 74 tons 8 cwt. 0 qr. 10$\frac{2}{3}$ lb.
27. 35 yds.
28. £64 9*s*. 2*d*.
29. 3,466,145 cub. yds.
30. 2292$\frac{3}{8}$ cub. ft.;
173 tons 19 cwt. 1 qr. 27$\frac{7}{8}$ lb.
31. 6·3 in.
32. 1·43 pints.
33. 385 balls; volume of box = 36·06 cub. ft. (side of base $= 6(9 + \sqrt{3})$ in.)
34. 84.
35. 650.
36. 560.
37. 624.
38. 2051.
39. 165.
40. 12.

CHAPTER V.

Section I. (Page 166.)

1. 82½ sq. ft.
2. 145 ft. (very nearly.)
3. 86·58321 sq. in.
4. 45 sq. yds. 2 ft.
5. 20 ft.
6. 3432 sq. ft.
7. 10½ in.
8. 5 ft. 6 in.
9. 3 ft. 2$\frac{2}{11}$ in.
10. 94·248 sq. in.; the same.
11. 9 ft. 4 in.
12. 6 sq. ft. 16 in.
13. 75$\frac{3}{7}$ sq. in.
14. 1566 sq. yds. 1 ft. 102$\frac{4}{7}$ in.
15. 62 sq. ft. 52$\frac{4}{11}$ in.
16. 4 yds.
17. 1 rood.
18. 161 sq. yds. 3 ft.
19. £29 6*s*. 8*d*.
20. 9 ft.
21. 2 hrs. 42 min.
22. 176 sq. yds.
23. 36·3247 sq. in.
24. 812 sq. yds.; £45 13*s*. 6*d*.
25. 24 sq. ft. 78 in.

Section II. (Page 174.)

1. 8 cub. yds. 9 ft. 50 in.
($= 123{,}725\pi$ cub. in.)
2. 13,577$\frac{1}{7}$ cub. yds. (4320π.)
3. 20 ft.
4. 15 sq. yds. 4 ft. 4 in.
5. 87 cub. ft. 521$\frac{1}{7}$ in. ($48{,}000\pi$.)
6. 28·2744 cub. ft.
7. 3 ft. 4 in.
8. 30 cub. ft. 1023 in.
9. 540 cub. ft. (nearly.)
10. 50 lb.
11. 625,571 yds.
12. 9127 in.
13. 14 tons 10 cwt. 1 qr. 25½ lb.;
£18 3*s*. 1*d*.
14. 9*s*. 2·88*d*.
15. 688¼ gallons.
16. 54 times.
17. £1 4*s*. 0¾*d*.
18. 2938 gallons.

19. 59 tns. 1 cwt. 2 qr. 15 lb. 12 oz.
20. 16 in.
21. 6 ft.; £10 14*s.* 6*d.*
22. 10 ft.
23. 12 (a little over).
24. 1·54 oz.; 2·24 oz.
25. 84,800 gallons.
26. 7 miles (nearly). (6·96 m.)
27. 6·38 in. per hour.
28. 2·32 min.
29. £335 2*s.* 6$\frac{10}{11}$*d.*
30. 132 sq. in.
31. 37 lb. 8 oz.
32. £10 11*s.* 6*d.*; 7920.
33. 276 lb.
34. 103 lb. 2 oz.
35. 6 cwt. 0 qr. 15$\frac{1}{2}$ lb.
36. £4752.
37. 589$\frac{2}{7}$ cub. in.
38. £4 1*s.* 2*d.*
39. 9·54 tons.
40. 465 cub. yds.

CHAPTER VI.

Section I. (Page 181.)

1. 4 ft. 6 in.
2. 38 sq. ft. 72 in. $\left(\frac{49\pi}{4}\text{ sq. ft.}\right)$
3. 314·16 sq. ft. (100π.)
4. 2·7258 ft.
5. 2·7614 ft.
6. 42 sq. ft. 96$\frac{2}{7}$ in.
   (= 1955π sq. in.)
7. 28$\frac{2}{7}$ sq. ft. (9π.)
8. 189·6113 sq. ft.
9. 54 sq. yds. 4 ft. 66$\frac{2}{7}$ in.
10. 13 in.
11. 3 sq. ft. 58·09 in.
12. 50 ft. 10 in.
13. 8 ft. 9 in.
14. 707$\frac{1}{7}$ sq. ft. (225π.)
15. 1634 sq. yds. 2$\frac{4}{7}$ ft.
    (4680π sq. ft.)
16. 4·7124 sq. ft.
17. 28·28 ft.
18. 7$\frac{1}{4}$ min.
19. 23 yds. 1 ft. 1$\frac{1}{2}$ in.
20. 87·7 yds.
21. £981 15*s.*
22. £202 11*s.* 5$\frac{3}{4}$*d.*
23. 70 cwt.; £73 10*s.*
24. 65 sq. in.
25. 38$\frac{1}{2}$ sq. in.; 9$\frac{5}{8}$ sq. in.
26. 231 acres.
27. 600 ft.
28. 117·81 sq. ft.
29. 18,844 sq. in.
30. 3·795 ft.

Section II. (Page 191.)

1. 16 cub. ft. 991$\frac{3}{7}$ in.
2. 2 cub. yds. 24 ft. 384 in.
3. 1232 cub. in.
4. 3142 cub. ft.
5. 8·196 ft.
6. 11·2 ft.
7. 31·416 ft. (10π.)
8. 20 ft. (very little over.)
9. 410$\frac{2}{3}$ sq. ft.
10. 2 sq. ft. 139$\frac{3}{7}$ in. (136π in.);
    1005$\frac{5}{7}$ cub. in. (320π.)
11. 6372$\frac{13}{23}$ cub. in.
12. 3 cub. yds. 21 ft. 1064 in.
13. 44 cub. in.
14. 90 sq. yds. 7$\frac{1}{7}$ ft.
15. 141$\frac{3}{7}$ cub. in. (45π.)
16. 713 cub. ft.
17. 192 cub. yds.

18. 382 tons.
19. 4·44 pints.
20. 1·46 oz.
21. £462.
22. 53,709 sq. ft.
23. ·131*d.*
24. ·0806 pint.
25. 1 cwt. 3 qr. 15 lb.
26. 4·88 in.
27. More than 4. (4·3425 exactly.)
28. 40 times.
29. 45 ft.
30. 48,977 sq. ft. (15,590π); 42,310 cub. yds. (363,624π cub. ft.)
31. 308 cub. ft.
32. $9\frac{1}{2}$ galls. (about.) (Between $9\frac{3}{8}$ and $9\frac{5}{8}$ galls.)
33. 36 galls. (about.) (Between $35\frac{3}{11}$ and $37\frac{1}{3}$ galls.)
34. 14 galls. (about.) (Between $13\frac{1}{3}$ and 15 galls.)
36. 986·8066 cub. ft.
37. 328·4328 cub. in.
38. 77·561 ft.
39. 235·96 cub. ft.
40. 194·2578 cub. ft.

CHAPTER VII.

Section I. (Page 204.)

(a)

1. 8 sq. ft. 104·64 in.
2. 4 sq. ft. 40 in.
3. 4 ft. 8 in.
4. 7 ft. 4 in.
5. 50·266 sq. ft.
6. 6 a. 2376 sq. yds.
7. £53 18*s.*
8. 7*s.* 6*d.*
9. £30 16*s.*
10. 195,355,200 sq. miles.
11. 183 sq. in.
12. 63·6174 sq. in.

(b)

13. 94·248 sq. ft.
14. 30 ft.
15. 10 ft.
16. 1 sq. ft. 10 in.
17. 1 sq. ft. 10 in.
18. $1489\frac{5}{7}$ (474π) sq. ft. *or* $7775\frac{3}{7}$ (2474π) sq. ft.
19. 27 sq. ft. $106\frac{4}{7}$ in. (1271π sq. in.)
20. 11 sq. ft. $75\frac{3}{7}$ in.
21. 3 sq. ft. 104 in.
22. $103\frac{5}{7}$ sq. yds.
23. 5 ft. 4 in.
24. 2403 sq. ft.
25. $\frac{1}{42,242}$.
26. 1050 ft.
27. 24 ft.
28. 8 ft. 4 in.
29. $\frac{1000}{2009}$.
30. 1,489,348 sq. miles.
(The altitude of the segment will be found to $=$ diameter $\times \sin^2 5^\circ$.)

Section II. (Page 212.)

(a)

1. 1 cub. ft. 1327 in. ($1\frac{9}{16}\pi$ cub. ft.); 6·2 ft.
2. 6 cub. ft. $1130\frac{2}{3}$ in.
3. 88 yds.
4. $68\frac{4}{7}$ sq. in.; 1000 cub. yds. (nearly.) (318·3π.)
5. 523·6 cub. ft.

6. (1) 7·44 in. (2) 3·39 in.
7. 11·6 in.
8. 17 lb. 3 oz.
9. $53\frac{2}{3}$ yds.
10. 1048 cub. ft.
11. £15,155.
12. Gold = ·248 sq. ft.; copper = ·416 sq. ft.; copper by 24 sq. in.
13. 28 lb. $10\frac{1}{3}$ oz.
14. 2 cwt. 18 lb. $10\frac{3}{14}$ oz.
15. 6 cwt. 1 qr. $9\frac{1}{2}$ lb.
16. $8\frac{209}{315}$ lb.
17. $95\frac{19}{35}$ lb.
18. 10 in.
19. 64 : 37.
20. 2·795 in.
21. 79·587 cub. in.; 22 lb. 6·141 oz.
22. 14 lb. 7 oz.
23. 1 ton 4 lb.
24. 66 lb. $10\frac{1}{2}$ oz.
25. 159 lb. $1\frac{6}{7}$ oz.
26. ·619 in.
27. (1) ·52 in. (2) ·828 in.

(b)

28. 10 cub. ft. 405·6 in.
29. 7068·6 cub. in.
30. 792 cub. in.
31. 6729 cub. ft.
32. 523·6 cub. in.
33. 1 cub. ft. 417 in.
34. 872 cub. ft. 756 in.; greater.
35. $84\frac{6}{7}$ cub. in.; $183\frac{1}{3}$ cub. in.
36. 2 cub. ft. 106 in. (approx.)
37. 3 cub. ft. 1435 in.
38. ·37267 pint.
39. ·235 lb.
40. ·88 lb.

CHAPTER VIII.

Section I. (Page 216.)

1. 4526·082 sq. ft.
2. 45·32 in.
3. 3 ft. 8·6 in.
7. 37·4 ft.
8. 41 sq. yds. 2 ft. 5 in.
9. 25 : 144.
10. 7·03 in.
11. 27 sq. ft.
12. 27 sq. ft.
13. 1 : 3 : 5 : 7.
14. 2·41 in.
15. 1 : 4.
16. 10·39 in.
17. 73 sq. ft.
18. 19*s.* $1\frac{1}{2}$*d.*
19. £455 0*s.* 4*d.*
20. 888 sq. yds. 8 ft.
21. £7 9*s.* 4*d.*
22. $28\frac{1}{2}$ sq. ft.
23. 156,025 : 11,664 *or* 13·38 : 1; 4095 miles (nearly).

Section II. (Page 224.)

1. 5 cub. yds. 21 ft. 663 in.
2. 8 ft.
3. 14·7 times.
4. £20 0*s.* $4\frac{1}{3}$*d.*
5. 1080 lb.
6. 17·2 ft.
7. 3 cub. yds. 1323 in.
8. $2\frac{1}{24}$ cub. in., $128\frac{5}{8}$ cub. in.
   Surface of pyr. 10·7 sq. in.
   „ frust. 166·7 sq. in.
9. 100 cub. in., 182·84 cub. in., 236·78 cub. in.
10. 1·65 ft. above the base.
12. 11·447 ft.

13. 5·39 : 1.
14. Altitude is divided in proportion 1 : ·26 : ·18. (50 : 13 : 9, nearly.)
15. 5 ft. 6 in.
16. 5 in.
17. 16 in.; 21·3 in.
18. £5 9*s*. $1\frac{1}{4}$*d*
19. 15·205 in.
20. 1920 gallons.
22. 54 : 53.
23. £41 12*s*.
24. $15\frac{3}{8}$ times.
25. £29 14*s*.
26. 5·6 in. (very nearly.)
27. $353\frac{1}{3}$ times.
28. 92,160 cub. ft.
29. 2292 cub. ft.
30. 3,466,145 cub. yds.

CHAPTER IX. (Page 229.)

(a)

2. 1·084 : 1. ($4^{\frac{1}{3}} : \pi^{\frac{1}{3}}$.)
3. 30 ft.
4. Equal; each 11·23 cub. in.
5. 1 ft.
6. The conical by ·242 pint.
7. $15\frac{3}{11}$ ft.
8. $9\frac{5}{8}$ sq. in.
9. By 1500 sq. ft.
10. 9·425 ft.
11. 9 ft. $0\frac{3}{4}$ in.
18. 24 cub. ft. short.
19. 1568 lb.
20. 10·2 in.
21. 1·47 times.
22. 6·9 in.

(b)

24. 10·9 min.
25. 3 ft. 7 in.; 175,968 galls.
26. 21 hrs. 38 min.
27. 15·3 miles.
28. 12·9 min.
29. 10,190 tons.
30. 1 ft. $2\frac{1}{3}$ in.
31. 45 times.
32. 100 (exactly).
33. 16 (exactly).
34. 101 times.
35. 1 ft. 6 in.
36. 1 ft. $2\frac{1}{2}$ in.
37. 1·16 lb.
38. 32.
39. ·304 in.
40. 9·87 in.
41. 675; ·432 oz. each.
43. 1765 ft. 3 in.
44. 33 lb. $13\frac{2}{7}$ oz.
45. 4 ft.
46. 4584 yds.
47. 1944 yds.
48. 20·8 in.

(c)

49. 3·7 oz.
50. 37 cwt. 0 qr. 20 lb. 1 oz.
51. 2513 cub. ft. (800π.)
52. 509 cub. ft. (162π.)
53. 379 sq. yds. (to nearest yd.)
54. 3036·88 sq. yds.
55. 7372 cub. ft.
56. 989 sq. yds. 3 ft.
57. 3500 sq. ft.
58. £6 10*s*. 8*d*.; 2511 cub. ft.
59. 3185·1 sq. ft.
60. 9 ft. 11 in.; 19 ft. 10 in.; 29 ft. 9 in.
61. 3,469,714 cub. ft.; by 333 cub. ft.
62. 11 cwt. 0 qr. $26\frac{1}{3}$ lb.
63. 44 ft. $5\frac{43}{77}$ in.
64. $1,288,010\frac{10}{49}$ tons.

65. 17 lb. 5 oz.
66. 24 cub. in.
67. 62·118 sq. in.
68. 9 lb. 7 oz.
69. 813 cub. in.
70. 33 cub. ft.
71. ·0067 pint.
72. 17 cwt. 2 qr. $3\frac{1}{2}$ lbs.

(N.B.—The solid is only partly immersed.)

(d)

73. 21 cub. ft.
74. 25·1328 sq. ft.
75. 1 sq. ft. 7 in.
76. 234 cub. in.
77. 3 sq. ft. 118 in.
78. 578 cub. in.
79. ·17 of the cone.
80. 21 cub. in.
81. 5 cwt. 1 qr. 11 lb.
82. 2 cwt. 24 lb. 1 oz.
83. 2·91 in.
84. 21·8 in.
85. 4·1 in. (4·2 if π be taken accurately.)
86. 2 tons 18 cwt. $\left(\frac{4131}{2}\pi \text{ lb.}\right)$
87. 573 cub. in.
88. 2 in.
89. 83·7 cub. in.; 151 sq. in.
90. 10·210 cub. in.
91. 7·3 cub. in.
92. 7·65 in.
93. 1 cub. ft. 82 in. $\left(\frac{\pi}{3} \text{ cub. ft.}\right)$
94. 698 cub. in.
95. 41 sq. ft. 128 in. $\left(\frac{40\pi}{3} \text{ sq. ft.}\right)$
96. $1\frac{1}{2}$ in., $\frac{3}{8}$.
97. ·22 in.
98. 1·6 in.
99. 6·8068 cub. in. $\left(\frac{13\pi}{6}.\right)$
100. $\frac{2}{\sqrt{3}}$ in.; ·174 cub. in.
101. $56\frac{1}{2}$ sq. in. (about). (18π.)
102. $\sqrt{2}$ in.; $\frac{88}{105}$ cub. in.